SECTION: ARTICLES

Discrete numerical estimation: a comparison between children and adults*

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Abstract

Numerical estimation is little studied in Brazil, although it has already a considerable international literature. However, most of these studies address the estimation ability of children, disregarding the higher stages of development of this ability when compared to adults. In this perspective, this study aims to compare the performance in discrete numerical estimation, for different forms of presentation of stimuli, between children, attending the 2nd to 6th grades (from public and private schools) and adult students in upper secondary education (Proeja) and higher education in mathematics. For this, we carried out a quantitative cross-sectional study based on the calculation of the relative accuracy presented by the students in a Discrete Numerical Estimation Test (DNET). The results indicated that 2nd and 3rd graders estimation skills are comparable to upper secondary adult students in most tasks. However, higher education students performed better in all tasks than the other students did. These results suggest that discrete numerical estimation is a skill that can be developed and improved throughout life, even in adulthood. In Mathematics Education, both children and adults are usually not accurate in their estimates, even though this ability is recognized as important for symbolic mathematical performance, based on the knowledge of number magnitudes.

Keywords

Discrete numerical estimation - Mathematical development - Estimation in adults.

Keywords

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Introduction

Historically, mathematics has been concerned with solving practical problems, especially to meet the needs of everyday problems that require faster answers rather than exact ones. However, when analyzing expressions such as mathematically, accurately or precisely, which emphasize accuracy as an essential attribute, speaking of estimation in a mathematical context may represent a contradiction. In realizing this need, society has been changing educational goals, among which reading, writing and calculating are no longer sufficient, focusing on the development of mental calculation and, eventually, of estimation. In this study, we aimed to verify at what stage of development the numerical estimate of children from the 2nd to the 6th school year is comparable to that of adults.

Although numerical estimation can be discussed in several respects (for example, approximate calculations, location of numbers on number lines, comparisons between quantities), this study focuses on the development of the Discrete Numerical Estimation ability (DNE) from the earliest years of elementary school to higher education. That is, our focus is on the ability to identify a cardinal number that represents a certain quantity of elements, without actually counting them. We chose this type of task because it reduces the amount of specific knowledge required, other than of quantification itself, thus facilitating its analysis in terms of the mathematical development of the individuals. Moreover, DNE tasks are judged more reliable than other numeric estimation tasks due to the familiarity of children, as well as of adults, with quantification tasks. A common learning activity of numerical representation in Brazilian schools is asking students to compare numbers with the quantities they express, since the first school years.

In view of the above, this article aims to identify and analyze: a) at which point in their development children's estimation accuracy is comparable to that of adults; and b) for which tasks (different scales, different densities, with or without reference matrix) children's estimation accuracy is comparable to that of adults.

In this study, adults are Upper Secondary Education students of the *Proeja* modality (National Program for the Integration of Basic and Professional Education in the Modality of Youth and Adult Education) and undergraduate students of Mathematics. All of them are at least 18 years old, according to the legal definition in force in Brazil, without specifying a maximum age. Children are defined according to the Brazilian *Child and Youth Statute* (ECA) (BRASIL, 1990), which establishes that adolescence begins at the age of 12, the maximum age of our sample of students from 2nd to 6th year, most of whom are between 7 and 12 years of age (exceptions are older students who are outside the age range stipulated for their school year). From now on, they will be referred simply as adults and children.

Considering that the quantities to be estimated were presented in different ways to the subjects, we hypothesized that, for all different modes of presentation, older individuals would be more precise than the younger ones, as demonstrated by accuracy levels in school tasks. Given the formal mathematical skills already developed in adulthood, adult students in upper secondary and higher education would perform similarly and comparable to 6th grade students, since it is in this period that literature shows evidence of children already having the skills to make accurate estimates.

This study is part of a more comprehensive project entitled *Diversity in learning of initial mathematics: the comprehension of numerical estimation* (Brazil Platform and Research Ethics Committee of the Federal University of Rio Grande do Sul under number 31575913.6.0000.5347). The goal of this project is to study diversity in mathematics learning, in terms of numerical estimation in different groups of students.

Theoretical framework

In order to be mathematically competent, in addition to understanding relations between quantities, it is necessary to master symbolic mathematics, beginning with numbers and their relations, the basic function of numbers being the representation of these quantities. The literature on the subject suggests a significant relationship between the ability to estimate and later mathematical knowledge, especially regarding arithmetic skills (BOOTH; SIEGLER, 2006; SCHNEIDER; GRABNER; PAETSCH, 2009; SIEGLER; MU, 2008). Symbolic mathematics is an exclusive human capacity. However, even if entirely abstract, estimating does not depend on the learning of a symbolic system, although it can extend to increasingly accurate judgments when a symbolic system is apprehended (ANTELL; KEATING, 1983; STARKEY; COOPER, 1980; XU; SPELKE, 2000).

To exemplify the relationship between mathematical knowledge and numerical estimation, Reys (1986) observed that children who estimated with greater accuracy had a better comprehension of place value, mental computation skills, error tolerance, understanding of arithmetic properties, confidence in their responses and variability in the use of strategies. Levine (1982) noted that quantitative, reasoning and computational skills were also strongly related to the ability to estimate. Besides these reasons, the importance of estimation is exemplified by its continuous and daily use. Knowing how to estimate requires going beyond the mechanical application of procedures, calling for flexible and contextual strategies appropriate to each situation presented.

In the same perspective, Siegler and Booth (2005) point out that some studies of difficulties in numerical estimation point to limitations of conceptual comprehension, of certain skills (such as counting and arithmetic) and of working memory. That is, children and adults who estimate accurately tend to have better conceptual understanding, better counting and arithmetic skills and greater working memory capacity than those who estimate with less precision.

However, it is a consensus among researchers that people do not estimate accurately. One of the reasons for weak estimation skills may be the diversity of tasks involved. For example, estimating the population of a country and calculating the approximate product of two factors or the speed of a moving car have little in common, except for the fact that the answer is given by estimation (BOOTH; SIEGLER, 2006). This variety also extends to the number of possible estimation strategies, to the tasks' difficulty levels and to the development patterns described in the literature.

Research on numerical estimation have increased in the last three decades, but in the 1980s, 90s and early 2000s they were aimed at understanding the process of comparison between quantities, of identifying sets with more or less items (STARKEY;

COOPER, 1980; ANTELL; KEATING, 1983; WYNN, 1992; DEHAENE, 1997; XU; SPELKE, 2000).

Studies of this type involve skills other than numerical estimation, which may be strictly related to perceptual ability in a purely continuous and non-discrete observation of elements (density, for example).

Few studies have discussed the numerical estimation of items considering the numerical magnitude of the sets (LUWEL et al., 2000; HUNTLEYFENNER, 2001; LEMAIRE; LECACHEUR, 2007; OBERSTEINER et al., 2014). Recent studies of numerical estimation have been discussing the estimation of the location of numbers on number lines (BOOTH; SIEGLER, 2006; EBERSBACH et al., 2008; PEETERS; VERSCHAFFEL; LUWEL, 2017), which is also related to numerical skills but differs in terms of the necessary requirements for its realization. Thus, the study of discrete numerical estimation is a little explored field, especially in recent years.

The literature presents Number Sense (NS) as a capacity related to people's numerical ability. Thus, numerical estimation should be considered an integral part of NS. However, there is no consensus on NS origin: neuropsychology indicates that NS is an innate capacity (DEHAENE, 1997); on the contrary, the constructivist perspective affirms that NS is built and progressively, internally organized (CORSO; DORNELES, 2012).

The ability to perform DNE goes beyond NS, as proposed by Dehaene (1997), and develops with age according to educational and cultural experience, advancing together with the development of the symbolic system (HALBERDA; FEIGENSON, 2008). This happens because the way of quantifying items based on writing or speech is quite different between different cultural groups (OPFER; SIEGLER, 2012) and that it provides the basis for most of the formal mathematical thinking.

In his study of numerical estimation development, Huntley-Fenner (2001) evaluated the ability of 15 children (5- to 7-years-old) and adults to estimate quantities of items (5, 7, 9 and 11) randomly distributed (40 times each quantity). The results indicated that children and adults have similar estimating abilities; the only difference found is that standard deviation of estimates is lower for adults. In a study in which children estimated quantities of objects in a container, Booth and Siegler (2006) found that first graders were less accurate than second graders, which was already expected. However, in contrast, fourth graders' estimates were less accurate than third graders' estimates. This result contradicts previous findings that numerical estimates get more accurate with age.

Obersteiner et al. (2014) made an important finding in their study comparing the performance of 202 first-grade children asked to estimate quantities between 1 and 20 dots. They observed that the number of items was a strong predictor of both the response time and the accuracy rates in the task when presented at random, but not when presented in a grid. However, the performance in this second task correlated with performance in an arithmetic test, even when other cognitive variables were controlled.

Regarding the effects of age and grid size on the accuracy of the estimation, Luwel et al. (2000) concluded that the accuracy of the sixth grade students' estimates was significantly greater than that of second grade students. Surprisingly, they observed no effect of the grid size, as well as of the interaction of age and grid size. When Lemaire and

Lecacheur (2007) assessed numerical estimation of adults and the elderly, they observed that both age groups presented comparable performance and that there were no age-related differences, with the exception that older participants took longer than young adults to provide their estimates. The authors suggested that numerical estimation tasks might involve specific cognitive processes that are pre-symbolic and invariant in adulthood.

Method

We adopted a quantitative approach in this study, in order to compare the interaction of the variables, considering the sample groups. A total of 730 subjects, children and adults, students of public institutions and of a private school located in the metropolitan region of Porto Alegre, RS, participated in the study. We selected schools that have students in the grades relevant to the research (second to sixth grades of elementary school and adults in upper secondary and higher education).

We chose second graders as the lower age limit in our children sample because at this stage they already know to count in hundreds, being able to recognize and understand numerical magnitudes up to 100. The upper limit (sixth grade) was chosen based on the hypothesis that, at this stage, children would already have enough mathematical knowledge to estimate with the same accuracy of adults. We chose a sample of adults still studying to avoid possible effects of the time out of school in the ability to make estimates.

Sample

We carried out a cross-sectional study with a sample of 730 students from the 2nd (N = 116), 3rd (N = 127), 4th (N = 94), 5th (N = 163) and 6th (N=176) grades and adult students of upper secondary education (Proeja - N = 29) and higher education in mathematics (N = 25). These students attended three schools: a municipal public school (N = 389) and a private school (N = 287), where studied the children in our sample, both in the city of Porto Alegre, RS, and a federal public institution offering upper secondary and higher education (N = 54), where studied the adults, in metropolitan region of Porto Alegre, RS.

Data collecting tool

As there are no reports of a test that evaluates the performance in numerical estimation tasks according to the individual relative precision (RP) and considering the same variables of this study, we structured and organized the instrument to collect data for our study. The Discrete Numerical Estimation Test (DNET) contains 64 tasks and evaluates the subject's ability to perform numerical estimation of items in a discrete set, taking into account the time required to prevent an actual counting of the items. In general, the tasks consist in counting dots in sets of different sizes, presented in the same color (black) and size.

In order to achieve our goals, we judged necessary to include different numerical scales in the study, represented by grids of different sizes (10x1, 10x2 and 10x10), assuming they represent different levels of difficult and considering that the ability to estimate is procedural and develops gradually. In the reviewed literature, there are indications that the estimation strategy and accuracy vary with the presentation format, with the fact of knowing or not knowing the maximum number of points in each grid (range of possible values) and with the distribution of the items in the grid, which may present greater (more thickly clustered dots) or lower (more spaced dots) density. These specific instrument variables are detailed below:

(a) Scale: One way to represent quantities is to use grids (arrays) that organize items into groups. The scale 10 (S10) corresponds to a grid with a single line, containing 10 white squares (10x1 grid); the scale 20 (S20) corresponds to a grid with two of these lines (10x2 grid); and finally the scale 100 (S100) is a grid with 10 lines, totaling 100 white squares. These grids are filled with black circles (only one per white square), distributed in these spaces. The items were also randomly presented (R) without a grid. That is, the same number of black circles was presented to the participants just scattered on the projected screen, without being distributed in a grid of squares for reference.

(b) Density: another way of evaluating the accuracy of the estimates was varying the distribution of the items in the grid. That is, the same quantities were presented in two ways: with clustered items (C) side by side, filling the empty squares sequentially, without leaving empty spaces between them, until completing the total amount of items to be estimated; or with spaced items (S), so that they are randomly distributed in the matrix, which presents empty squares amid others filled with dots.

(c) Knowledge of the maximum value: also important to understand the estimation process, we tested if knowing the maximum number of dots in the grid (MK) would lead to more accurate estimates when compared to the estimates made when this number was unknown (MU), even with dots distributed in a fixed-size grid. That is, at first the students were asked to estimate the quantities not knowing the maximum number of dots that could fit in the matrix, since no information was given on the amount of white squares. In a second moment, the students were asked to estimate these same quantities knowing that the maximum number of dots coincided with the number of white squares in the grid.

All variables interacted with each other. For example, on the scale 10 (S10) the quantities of 4 and 7 dots were presented in a clustered or spaced manner in the grid. At first, the students did not know the information about the maximum number of dots that could fit in the grid. In a second moment, they were asked to estimate the same quantities; this time, however, knowing how many dots could fit in the grid. Likewise, students estimated quantities 4, 7, 9 and 17 in a 10x2 grid on the S20 scale. Finally, they were asked to estimate quantities 4, 7, 9, 17, 25, 49, 78 and 95 in a 10x10 grid (S100) under the same conditions presented previously. As the last task of the instrument, the subjects estimated the same quantities proposed for S100, but with dots arranged randomly on the screen and not in checkered grids as in the other tasks and scales.

We assessed the applicability of this instrument based on the conclusions of a preliminary study (DORNELES et al., 2015), which enabled the necessary adjustments to make this instrument a more reliable tool for data collection.

Data collection

The procedure for conducting the DNET consisted of evaluating the students together with their classmates, so that the activities could be performed in a work environment familiar to them. The mean time of the test was 50 minutes per class. The tasks of the DNET were presented visually, with the aid of a multimedia projector. Afterwards, the participants wrote down in an answer book their estimation for the presented image.

The display time was adjusted for each number of items so that there was sufficient time for participants to observe the quantities but not for counting. Each new quantity was only presented after all participants confirmed that they had already completed the previous task, ensuring that only a small percentage of the data was lost due to lapses of attention. There was no feedback on the answers.

Data analysis

To evaluate the ability to perform numerical estimation of quantities, the statistical analysis of the data considered the variable *relative precision* (RP) in the different scales (S10, S20 and S100), grades (2nd, 3rd, 4th, 5th, Proeja-P or higher education-H), schools (public-Pu or private-Pr), while knowing the maximum number of dots in each grid (maximum known-MK) or not knowing it (maximum unknown-MU), in a clustered (C) or spaced distribution (S) of the dots in the grids and in a random distribution (R), without an auxiliary grid.

We used a formula adapted from Siegler and Booth (2004) to calculate the Relative Accuracy (RA), considering the *True Value* the actual quantity to be estimated and *Estimate* the quantity determined by the subject when performing the task: $RA = (|REAL\ VALUE - ESTIMATE|)/(SCALE\ (10{,}20 \text{ ou } 100))$. For exemple, if the true value is 95 and the subject's estimate is 97, in scale S100, the relative accuracy is $RA=(|95-97|)/100 = 2/100= 0{,}02$. The closer to zero is the result for RA, the more accurate is the estimate.

We used an alpha confidence level of 0.05 (5% error probability) for the statistical tests. The exact p-values were reported and, when very small, were rounded to $p <0.001$. The Shapiro-Wilk normality test made it possible to verify the distribution of RA. Observing an asymmetric distribution, we chose to use the non-parametric *Mann-Whitney* test, which found a significant difference between different schools. Therefore, the subsequent analyzes considered separately the results by school. There was no significant difference for sex at any of the scales.

We used the Generalized Estimating Equation (GEE) procedure to analyze the relation between the variables, since the same subject answered several questions of the same test. The GEE procedure is appropriate for data correlated over time but showing a non-normal distribution. In this case, the GEE considers each of the 64 estimates made in the tasks of the DNET as related to the same subject and not to 64 different subjects, which would represent 46,720 responses. The Correlation Matrix tool enables us to calculate the correlation between variables and, among the possibilities of a working correlation matrix, we used an exchangeable correlation matrix and a robust estimator covariance matrix, considering the two most used tests in the literature for asymmetric variables and for a normal distribution with identity function.

We chose to use the normal distribution (even with the variable not being symmetric) and not the gamma distribution (logarithmic) distribution because the cases in which the individual hits the true value (i.e. in which their precision is zero) would be excluded, since the logarithm of zero is not defined. The post hoc analysis used was the multiple comparison test of Bonferroni and not the LSD test because the latter is very sensitive even to small differences, even if they are not relevant. ANOVA showed a difference between groups, but did not specified which groups differ. The Bonferroni post hoc test thus allowed each test to be performed at a significance level of at least 1-α, adjusting the confidence level for each individual interval, in this case, α = 0.05.

Results

The factor analysis of the variables maximum known or unknown (MK/MU); clustered, spaced or random (C/S/R); grade (2nd to 6th, P and H); and public or private school (Pu/Pr), in the three different scales (S10, S20 and S100), with grade and school as fixed variables and comparing the students' mean in each of the scales was detailed in Table 1. The students of Proeja (P) and higher education (H) appear twice in the tables to facilitate comparing their performance with that of the children.

Table 1 - Relative Accuracy means per grade and school at different scales

Grade	School							
	Public				Private			
	S10	S20	S100	R	S10	S20	S100	R
2	.0867	.1240	.1419	.1441	.0588	.0626	.0768	.1211
3	**.0943**	.0949	.1177	**.1441**	.0490	.0463	.0687	**.0787**
4	.0796	.0852	.1166	.1177	.0447	**.0485**	.0631	.1120
5	.0418	.0509	.0846	**.0864**	.0345	.0411	.0591	**.1024**
6	.0324	**.0347**	.0702	**.0916**	.0304	**.0370**	**.0634**	.0913
P	**.0333**	**.0628**	**.0716**	.0803	.0333	**.0628**	**.0716**	.0803
S	.0215	.0307	.0301	.0677	.0215	.0307	.0301	.0677

Source: The authors (2016).

The data in the table above shows that the higher the student's level of education, the more accurate the student's estimate is, except in some cases and, mostly, in the case of Proeja students, which presented lower estimation accuracy than their peers, adults attending higher education courses. Table 2 shows the statistical differences between children and adults, in terms of RA mean values indicated in Table 1.

Table 2 - Significant differences between the Relative Accuracy means per grade and school at the different scales

Scale	Grade	P	p	H	p
S10	2	Pu	0.025	Pu	<0,001
				Pa	0,011
	3	Pu	0.027	Pu	<0,001
	4			Pu	0,028
S20	2			Pu	<0,001
				Pa	0,018
	3			Pu	<0,001
	4			Pu	0,027
S100	2	Pu	0.006	Pu	<0,001
				Pa	<0,001
	3			Pu	<0,001
				Pa	<0,001
	4			Pu	<0,001
				Pa	<0,001
	5			Pu	<0,001
				Pa	<0,001
	6			Pu	<0,001
				Pa	<0,001
R	2	Pu	<0.001	Pu	<0,001
		Pa	0.015	Pa	<0,001
	3	Pu	<0.001	Pu	<0,001
	4	Pu	0.049	Pu	0,001

Source: The authors (2016).

We verified the similarities in terms of accuracy among the groups of students of the different school levels. Proeja's students did not show a consistent pattern, with the mean RA of this group not differing from the others in most analyzes. Considering both schools, we observed that for the scale 100, higher education students were significantly more accurate than all the other students in the sample at all scales, which, surprisingly, did not occur for random items, except when compared to second graders.

In all cases, upper secondary Proeja students did not show a pattern of regularity. This may be due to the small number of participants or to the great diversity in terms of age and experience of the participants of this school level. In general, when compared to private school students, adults showed accuracy comparable to third graders. However, in the public school, only the sixth grade showed a degree of accuracy comparable to higher education students.

On the other hand, we observed that estimates of randomly distributed items (R) are equally inaccurate for all subjects, children and adults. This conclusion is further confirmed by the incompatibility of the means obtained by adult students of higher education (H) with those of adult students of the Proeja (P), who often had a performance comparable to students of the first grades of elementary education.

The comparison between the Maximum Known (MK) and Maximum Unknown (MU) variables for different Grades and Scales showed significant results at the scale 100, as described in Table 3.

Table 3 - Relative Accuracy means for Maximum Unknown (MU) and Maximum Known (MK) per school and grade at Scale 100

Grade	Public School		Private School	
	MU	MC	MU	MC
2	0.1465	0.1369	0.0936	0.0603
3	0.1172	0.1181	0.0851	0.0523
4	0.1406	0.088	0.0888	0.0374
5	0.1167	0.0519	0.0815	0.0371
6	0.0896	0.0511	0.0912	0.036
P	0.0756	0.0676	0.0756	0.0676
H	0.0369	0.0232	0.0369	0.0232

Source: The authors (2016).

The data presented shows that students of all grades differed from higher education students, including Proeja students, which are also all adults. That is, in the cases where the maximum number of dots is not known, higher education students are more accurate than all other private school subjects are. Higher education students' accuracy was significantly greater than the accuracy of other students in both schools, regardless of whether or not they know the maximum number of dots in the grid.

The comparison between the variables Clustered (C) and Spaced (E) for different Grades and Scales showed significant results obtained at the scale 100, as described in Table 4.

Table 4 - Relative Accuracy means for Clustered (C)/Spaced (S) items per grade and school at Scale 100

Grade	Public School		Private School	
	C	S	C	S
2	0.1319	0.1514	0.0626	0.0913
3	0.1049	0.1304	0.051	0.0866
4	0.0974	0.1328	0.0466	0.0795
5	0.0671	0.1023	0.0406	0.0779
6	0.0474	0.0936	0.0429	0.0838
P	0.0505	0.0926	0.0505	0.0925
H	0.0128	0.0474	0.0128	0.0474

Source: The authors (2016).

Higher education students showed greater accuracy in both forms of presentation of the dots. The accuracy of sixth graders differed significantly from that of higher education students in both situations.

Table 5 presents the comparison between the Clustered (C) and Spaced (S) and Maximum Known (MK) and Maximum Unknown (MU) variables for different Grades and Scales.

Table 5 - Relative Accuracy means in the interaction between the variables: Grade, C/S and MK/MU, in the different grades, schools and scales

Grase	Scale	Maximum Unknown				Maximum Known			
		Public		Private		Public		Private	
		C	S	C	S	C	S	C	S
2	E10	.0693	.0898	.0598	.0504	.0864	.1014	.0662	.0583
	E20	.1188	.1157	.0716	.0691	.1275	.1341	.0660	.0443
	E100	.1602	.1327	.1079	.0795	.1426	.1312	.0748	.0458
3	E10	.0587	.0870	.0406	.0510	.1054	.1303	.0604	.0442
	E20	.1051	.0916	.0632	.0458	.1025	.0809	.0516	.0247
	E100	.1361	.0984	.1057	.0648	.1246	.1115	.0675	.0371
4	E10	.0522	.0690	.0356	.0500	.1057	.0922	.0295	.0637
	E20	.0874	.0757	.0635	.0580	.0904	.0860	.0425	.0287
	E100	.1507	.1309	.1111	.0666	.1143	.0621	.0480	.0268
5	E10	.0422	.0481	.0335	.0476	.0368	.0410	.0285	.0283
	E20	.0642	.0553	.0559	.0383	.0499	.0340	.0455	.0248
	E100	.1395	.0938	.1050	.0581	.0640	.0399	.0512	.0230
6o	E10	.0339	.0470	.0294	.0407	.0235	.0250	.0204	.0312
	E20	.0445	.0389	.0508	.0477	.0326	.0232	.0286	.0212
	E100	.1210	.0582	.1213	.0607	.0658	.0365	.0467	.0252
P	E10	.0190	.0431	.0190	.0431	.0277	.0347	.0257	.0327
	E20	.0698	.0523	.0698	.0523	.0784	.0504	.0784	.0504
	E100	.1048	.0466	.1047	.0466	.0807	.0545	.0806	.0544
H	E10	.0200	.0520	.0200	.0520	.0080	.0060	.0080	.0060
	E20	.0505	.0176	.0505	.0176	.0440	.0100	.0440	.0100
	E100	.0550	.0192	.0550	.0192	.0400	.0063	.0400	.0063

Source: The authors (2016).

Table 6 shows the significant differences in mean relative accuracy of public and private school students, obtained in the triple interaction for scale 100. They are highlighted when differing from those of adults.

Table 6 - Significant differences between Relative Accuracy means in school grades, with fixed Clustered (C)/Spaced (S), Maximum Known (MK)/Maximum Unknown (MU) and School

C/S	MK/MU	GRADE		Public (p)	Private (p)
		2	P	<0.001	
			H	<0.001	<0.001
		3	P	0.001	
			H	<0.001	<0.001
		4	P	<0.001	
			H	<0.001	<0.001
		5	P	<0.001	
			H	<0.001	<0.001
		6	H	<0.001	<0.001
		2	H	<0.001	<0.001
		3	H	<0.001	<0.001
		4	H	<0.001	<0.001
		5	H	<0.001	<0.001
		6	H	0.001	0.001
		2	P	<0.001	
			H	<0.001	<0.001
		3	P	0.052	
			H	<0.001	<0.001
		4	P	0.002	
			H	<0.001	<0.001
		5	P	0.003	
			H	<0.001	<0.001
		6	H	<0.001	<0.001
		P	H	<0.001	<0.001
		2	H	<0.001	<0.001
		3	H	<0.001	0.009
		4	H	<0.001	
		5	H	0.008	

Source: The authors (2016).

It is noteworthy that for the private school, spaced items and maximum number of dots known, sixth graders do not differ in accuracy from higher education students, indicating that adults attending school are always much more accurate in most situations. The data indicate that children in both schools only reach the degree of accuracy presented by adults in higher education when items are presented in a spaced form in the grid with maximum number of dots known. For both schools, the degree of accuracy of the children only reached that of higher education adults at the scale 100, with items spaced and maximum known for private school fourth graders.

Discussion

Some authors attribute the fact that younger children estimate with lower accuracy, when compared to older ones, to the progressive change in mental representations of scale, assuming that, at a given moment, they reach the level of adult mental representation (EBERSBACH et al., 2008). In the research by Siegler and Opfer (2003), the data indicated that each child has multiple numerical representations and that increasing age and numerical experience and the numerical context influence the choice of representation model. On the other hand, Dehaene et al. (2008) showed that linear representation might not occur, even with adults, depending on the type of stimulus presented.

In our study, in general, the higher the student's level of schooling, the more accurate he or she is and, for both schools, the greater accuracy of higher education students stands out. The results shows that the children reached the degree of accuracy of higher education adults when the items were presented spaced in a grid of maximum number of dots known, at scale 100, and only from the fourth grade onwards in the private school.

Proeja adults presented varying accuracy in their estimates, rarely differing from students of other school grades, but often differing from higher education adult students. This result suggests that the differences presented by the groups of adult students may reflect differences related to schooling and familiarity with mathematics. Two elements can explain this fact: the small number and age diversity of the sample of these students, which can vary greatly in terms of experience. However, this difference found among adults (Proeja and higher education students) might be a consistent proof that the development of numerical estimation is continuous and can be developed and improved throughout life, with new experiences, as presented in the theoretical discussion section of this study.

The data of Siegler and Booth (2005) suggest that changes in mental representations of numbers, once made, remain stable over time and, in general, in relation to different tasks. Changes can occur, but do not alter the general structure of thought. In the study by Lemaire and Lecacheur (2007), adults and the elderly presented similar performance in discrete numerical estimation. This is not in contradiction to the findings of the present study, since our sample of Proeja students was very diversified in terms of age, which were similar for the two groups of adults surveyed. In this case, factors related to schooling and experience are present.

When comparing the different scales, we observed that public school fifth graders and private school third graders estimate as accurately as adults do. The data suggest that the scale 10 is easy enough for children and adults alike, while random presented items were more difficult for all. Scale 20 presented similar results. Only at the scale 100 did adults demonstrate greater accuracy with respect to children.

Thus, as we discussed above, Huntley-Fenner (2001) evaluated the ability of children and adults to estimate randomly distributed quantities. His results, as well as ours, indicated that children and adults are able to make similar estimates, but the standard deviation of estimates is lower for adults. The data make it clear that there are two types of number representation: discrete and accurate (requiring language and culturally derived) or analog and approximate (independent of language and common to a number of species). In this case, the author indicates that analog number representations of adults are similar to those of young children, both qualitatively and quantitatively. The results of Huntley-Fenner (2001) agree with the findings of our study for private school students' estimates at small scales. However, the data for scale 100 suggest that using more efficient estimating strategies, influenced by knowledge of the relationship between numbers, increases the accuracy of the estimates.

We found that all participants showed lower accuracy in the estimation of randomly distributed items (R). Knowing the maximum number of dots in the grid and the comparison between the Clustered (C) and Spaced (S) variables only correlated with a significant improvement in the accuracy of students' estimates at scale 100. However, higher education students were significantly more accurate than the others were, regardless of knowing or not the maximum number of dots in the grid and in both forms of presenting the dots (C or S). Obersteiner et al. (2014), in their study of numerical estimation by children, found that the number of items was a strong predictor of accuracy in the enumeration task with random arrangements, but not in the task with grids. In the case of the grid task, children can make use of strategies based not only on external representations of numbers but also on established relationships between numbers.

By analyzing the data on whether or not subjects know the maximum number of dots in the grid, we verified that this information is only important for the accuracy of the estimates in the scale 100, for which the adults' accuracy was higher than children's accuracy. However, in all cases adults showed greater accuracy. In the public school, accuracy at scale 100 was greater for sixth graders and higher education students in the tasks involving clustered dots. The results may also indicate that knowledge of relationships between numbers such as proportion and multiplication might contribute to improving the subjects' estimation, but not the other way around.

We conducted a four-way analysis of the variables density (C/S) and knowledge of the maximum number of dots in the grid (MK/MU) per year and per scale. It showed that children's degree of accuracy only reached that of higher education adults in scale 100, with spaced items and maximum known for private school fourth graders. The distributions of the children's estimates were different from those of adults. The data suggest that the analog number representation of the children is qualitatively distinct from that of adults.

Given the data, it is likely that children's estimates will follow a pattern similar to that of adults. However, since children make use of fewer estimation strategies (or use them less efficiently), their estimates are expected to be generally more variable than adults' estimates, and therefore different. Although all children and adults participating in this study have already begun formal instruction in mathematics, the results presented show that the accuracy in numerical estimation is still developing throughout this time.

Although this study's aim is not to compare the performance of public and private school students, it is important to highlight this fact, since these results may be related to factors such as educational practices or administrative arrangements, which may affect educational outcomes. Moreover, the choice of studying in a public or private school is determined primarily by the socioeconomic situation of the family. All of these factors may be decisive for the better performance of private school students when compared to their peers in public schools.

Final considerations

This study aimed to verify at what point in their development children acquire the degree of accuracy in numerical estimation similar to that of adults. Earlier studies already have shown that this ability is more accurate in older children than in younger ones.

The main hypothesis was that the ability to estimate increases qualitatively with age and experience, developing progressively until the moment when the acquisition of a formal knowledge enables achieving a degree of accuracy comparable to that of adults. This hypothesis was partially confirmed because, for all different modes of presentation of the stimuli, older subjects were more accurate than the younger ones (except for upper secondary adult students), and this development was demonstrated in similar age groups, but differing according to the way of presenting the dots. In tasks of greater difficulty, sixth graders and even adults in upper secondary education did not achieve accuracy comparable to those of higher education students, contrary to expectations. That is, the development is progressive, but distinct for each difficulty presented in the tasks.

The hypothesis that the children's estimation ability is comparable to that of the adults from the sixth grade onwards, when the obtainment of formal knowledge is expected, was drawn from evidence presented in the literature, according to studies of number line estimation. We expected adult estimates, irrespective of schooling, to be similar. However, the ability to estimates seems to evolve throughout life, even after achieving the ability to represent exact numbers.

The substantial differences between upper secondary and higher education adult students also suggest that these skills can develop not only with age, but also through experiences made possible by schooling, indicating that not only organic factors are related to this ability, but factors external to the individual as well. Also new is the finding that the estimation of children in the fifth and sixth grades may already be comparable to that of university students under specific conditions of presentation of the tasks. As far as we know, before ours no study compared the performance of these children with the performance of adults for different forms of presenting the tasks.

We observed a considerable difference between the accuracy obtained by public and private school students in the different tasks requested, indicating that factors such as socioeconomic level and cultural experience, for example, may interfere with this ability. Differences in classroom mathematics content and in mathematical knowledge may also have influenced the results, when comparing schools. These are complementary pieces of evidence that discrete numerical estimation is influenced by other abilities. In this case, we observed that there are different levels in the development of the estimation ability, but this development occurs in different years in different schools. The results suggest that these differences relate more to schooling than to age.

Science seeks an explanation, a better description of the world, and a possible generalization of results. Good research must be replicable in another context, being more descriptive than prescriptive. In this case, education uses a set of applied sciences that together need to arrive at answers about the development of human learning. Education has raised fundamental questions, but still needs answers that are more precise. There is a lot of discussion about new teaching methods, but almost no discussion about child development and learning. That is why studies of factors related to math learning are so important.

Levine (1982) and Reys (1986) have already observed in the 1980's that accuracy in numerical estimation is related to place value comprehension, mental calculation skills, tolerance to err, understanding of arithmetic properties, confidence in one's response and diversity in the use of strategies. Our study indicates the need for greater development of these skills in school, because, besides the reasons listed, numerical estimation ability goes beyond the mechanical application of procedures and require the elaboration of strategies appropriate to different situations.

Therefore, the fact that neither children nor adults are good at estimating highlights the inability of these students to develop strategies for estimating more accurately, indicating limitations in the conceptual understanding of numbers and in arithmetic ability (SIEGLER; BOOTH, 2005). The teaching of numerical estimation, however, has not been given due importance, even if this ability is so present in the daily life of children, and perhaps because of this, one may have the false idea that it is something easy to understand. In this sense, it is evident the need for further studies in this field.

This study aims to problematize the importance of stimulating the development of numerical estimation skills in children from the first school grades until adulthood. Mathematics teaching, on the contrary, has been based on calculation and algorithms rather than mathematical understanding. However, as the literature already makes clear, numerical estimation enables the development of strategies for solving complex mathematical problems and gives students the understanding of number magnitudes.

Our results help to increase the knowledge about the development of numerical estimation in children and adults, indicating the need to develop this ability in the classroom. Previous research suggests that estimation is not an independent skill, but that it seems to be related to other mathematical skills, such as mental computing, spatial visualization and measurement, for example. In this sense, further research may explore the association between discrete numerical estimation and other mathematical skills. Although this study provided important information about estimates, other questions remain unanswered, especially those related to the cognitive abilities involved in the different estimation strategies.

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