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Scaling Symmetries and Conservation Laws for Variable-coefficients Nonlinear Dispersive Equations

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ABSTRACT. Scaling symmetries arise in different branches of physics, and symmetry-based approaches are powerful tools for studying scaling-invariant models since they can provide conservation laws that are not obvious by inspection. In this framework, the class of variable-coefficients nonlinear dispersive equations vc$K(m,n)$, which contains several important evolution equations modeling nonlinear phenomena, is considered. For some of its scaling-invariant subclasses, we study its nonlinear self-adjointness and construct eight new local conservation laws associated with scaling symmetries by using a general theorem on conservation laws and the multipliers method. The property of scale invariance of those equations led to five conservation laws with a direct physical interpretation: energy, center of mass, and mass are the conserved quantities obtained in some cases.

Keywords: scaling symmetries, variable-coefficients nonlinear dispersive equations, nonlinear self-adjointness, conservation laws.

1 INTRODUCTION

Scaling symmetries have wide applications in science and in engineering and are far from being a special case in physics - they can be found, for instance, in quantum physics, fluid mechanics, turbulence, elasticity, heat diffusion, convection, filtration, gas dynamics, and also in the theory of detonation and combustion (see [12] and references therein). Besides, scaling invariance is closely related to the theory of fractals as well as to the general theory of dimensional analysis and renormalization [12]. By having these considerations in mind and motivated by remarkable features of the compacton $K(m,n)$, introduced by Rosenau and Hyman [35], Souza and Silva [37, 38] have recently employed the Lie symmetry machinery [9, 33, 39] to build up a generalized Rosenau-Hyman equation invariant under the scaling symmetry of standard KdV and obtained a variable-coefficients $K(m,n)$ (vc$K(m,n)$ hereafter) of the form

$$u_t + [f(t,x)u^m + g(t,x)(u^n)_{xx}]_x = 0\,, \quad m,n > 0\,, \tag{1.1}$$

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where $f(t,x) = \frac{c_1}{m} t x^{2m-7}$, $g(t,x) = \frac{c_2}{n} t x^{2n-5}$, c_1 and c_2 non-zero constants. In [38], the package SADE [34] for Maple was used to seek for scaling-invariant solutions for vc$K(2,2)$ and vc$K(3,3)$ equations and rational similarity solutions of type $u(t,x) = F(x/t)$ were found. The physical relevance of such solutions, also called self-similar solutions, is well known. All rational solutions can be obtained by examining solitonic-type solutions in an appropriate limit [17]. Additionally, self-similar solutions often describe the intermediate asymptotics of a problem, i.e., when a system has evolved into a range of time in which neither the initial data nor the boundary conditions effects dominate the solution [12]. From a mathematical point of view, similarity solutions are taken as a standard procedure for reducing partial differential equations to ordinary ones [17]. The vc$K(m,n)$ (1.1) is a class of variable-coefficients scaling-invariant nonlinear dispersive evolution equations which possess physically appealing solutions. Since several of its subclasses are potentially applicable for modeling nonlinear phenomena, it is surely worthy of further investigation.

Conservation laws are not only cornerstones of physics but also a relevant tool for studying the integrability of partial differential equations, as well as the existence, uniqueness, and stability of solutions. Nonlinear partial differential equations have been successfully employed to describe evolution of a sort of physical systems. However, since commonly evolution equations do not have an usual Lagrangian, it is not possible to associate conservation laws with their symmetries through the celebrated Noether's theorem [32]. Hence, one needs to seek for other approaches to build up conservation laws if dealing with non-variational problems. As reported in [31], there are several available routines to this end. To mention a few, Anco and Bluman proposed a direct algorithm, often referred as multipliers method, for constructing local conservation laws to partial differential equations expressed in a standard Cauchy-Kovalevskaya form [5, 6]. A partial Noether approach due to Kara and Mahomed has proven to be quite efficient for Euler-Lagrange-type equations [27], and there are also other methods that differ from the Noether's or the above mentioned ones based on Lax pairs of nonlinear evolution equations (see, e.g., [29] and references therein). Some years ago, Ibragimov presented a general theorem for constructing conservation laws based on the self-adjointness concept [23, 24]; later on, it was generalized to nonlinear self-adjointness [25, 26]. Concerning classes of third-order nonlinear evolution equations, there are several works devoted to classify them as nonlinearly self-adjoint and to construct conservation laws via Ibragimov's theorem [10, 11, 18, 19, 20, 21, 41, 42]. The relations between Ibragimov's approach and the direct method are well known [4, 46, 47], and the latter is also largely employed to build up conservation laws for nonlinearly self-adjoint equations [7, 13, 15, 22, 30, 44].

In this paper we seek for conservation laws associated with scaling symmetries for scaling-invariant subclasses of a generalized vc$K(m,n)$. To this end, we shall consider the expanded form of vc$K(m,n)$ (1.1),

$$\begin{aligned} &u_t + \alpha_0(t,x,u) + \alpha_1(t,x,u)u_x + \alpha_2(t,x,u)u_x^2 + \alpha_3(t,x,u)u_x^3 \\ &+ \alpha_4(t,x,u)u_x u_{xx} + \alpha_5(t,x,u)u_{xx} + \alpha_6(t,x,u)u_{xxx} = 0\,, \end{aligned} \tag{1.2}$$

with coefficients $\alpha_i(t,x,u)$, $i=0,1,\ldots,6$, written in terms of arbitrary $f(t,x)$ and $g(t,x)$, i.e.,

$$\begin{aligned}&\alpha_0 = u^m f_x\,,\ \alpha_1 = mu^{m-1}f\,,\ \alpha_2 = n(n-1)u^{n-2}g_x\,,\\ &\alpha_3 = n(n-1)(n-2)u^{n-3}g\,,\ \alpha_4 = 3n(n-1)u^{n-2}g\,,\\ &\alpha_5 = nu^{n-1}g_x\,,\ \alpha_6 = nu^{n-1}g\,.\end{aligned} \tag{1.3}$$

We study the nonlinear self-adjointness of the generalized vc$K(m,n)$ (1.2) and illustrate our result with some of its scaling-invariant subclasses, i.e., the time-dependent KdV, cylindrical KdV, time-dependent mKdV, time-dependent Schamel, and vc$K(m,n)$ (1.1). Hereafter, we construct local conservation laws associated with scaling symmetries for particular cases of these equations by employing a general theorem on conservation laws and the multipliers method. We highlight that conservation laws associated with scaling symmetries can be obtained via multipliers method by computing fluxes with no integration involved [3, 8, 14].

2 NONLINEAR SELF-ADJOINTNESS CLASSIFICATION

The partial differential equation $F(x,u,u_{(1)},\ldots,u_{(s)}) = 0$ we consider is the generalized vc$K(m,n)$ (1.2). According to Ibragimov's theorem [23, 24], we first write its formal Lagrangian as

$$\mathscr{L} = \left[u_t + \alpha_0 + \alpha_1 u_x + \alpha_2 u_x^2 + \alpha_3 u_x^3 + \alpha_4 u_x u_{xx} + \alpha_5 u_{xx} + \alpha_6 u_{xxx}\right] v\,, \tag{2.1}$$

where t and x are the independent variables, $u(t,x)$ and $v(t,x,u)$ the dependent variables, and α_i, $i=0,1,...,6$, the coefficients given by relations (1.3). The adjoint equation is obtained by

$$F^*(x,u,v,\ldots,u_{(s)},v_{(s)}) := \frac{\delta\mathscr{L}}{\delta u} = 0 \tag{2.2}$$

wherein s is the maximal order of derivatives,

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i\frac{\partial}{\partial u_i} + D_iD_j\frac{\partial}{\partial u_{ij}} - D_iD_jD_k\frac{\partial}{\partial u_{ijk}} + \ldots\ (i,j,k=1,2) \tag{2.3}$$

denotes the variational derivative, and

$$D_i = \frac{\partial}{\partial x^i} + u_i\frac{\partial}{\partial u} + v_i\frac{\partial}{\partial v} + u_{ij}\frac{\partial}{\partial u_j} + v_{ij}\frac{\partial}{\partial v_j}\cdots \tag{2.4}$$

is the total differentiation with respect to x^i $(x^1 = t, x^2 = x)$. In order to obtain the adjoint equation of vc$K(m,n)$ (1.2), we write

$$F^* = \frac{\partial\mathscr{L}}{\partial u} - D_t\frac{\partial\mathscr{L}}{\partial u_t} - D_x\frac{\partial\mathscr{L}}{\partial u_x} + D_x^2\frac{\partial\mathscr{L}}{\partial u_{xx}} - D_x^3\frac{\partial\mathscr{L}}{\partial u_{xxx}} = 0\,. \tag{2.5}$$

Hence, given (2.1) and (2.5), the adjoint equation reads as

$$F^* = v_t + (\alpha_1 + \alpha_{5_x})\,v_x + 2\alpha_5 v_{xx} + \alpha_6 v_{xxx} = 0\,, \tag{2.6}$$

wherein $\alpha_{5_x} = nu^{n-1}g_{xx}$ and the other coefficients are given by relations (1.3).

Definition 2.1. *A partial differential equation $F(x,u,u_{(1)},\dots,u_{(s)})=0$ is said to be nonlinearly self-adjoint [25,26] if the equation obtained from the adjoint equation (2.2), after the substitution of $v=\phi(t,x,u)$, $\phi\neq 0$, is identical to the original equation, i.e.,*

$$F^*|_{v=\phi(t,x,u)}=\lambda F\ , \tag{2.7}$$

with an undetermined coefficient λ.

We assume $v=\phi(t,x,u)$ and consider Eq. (2.6) such as $F^*|_{v=\phi(t,x,u)}=\lambda F$, F given by the l.h.s. of vc$K(m,n)$ (1.2), to obtain

$$\phi_t-\alpha_0\phi_u+(\alpha_1+\alpha_{5_x})\phi_x+2\alpha_5\phi_{xx}+\alpha_6\phi_{xxx}=0\ , \tag{2.8}$$

$$\alpha_2\phi_u-2\alpha_5\phi_{uu}-3\alpha_6\phi_{xuu}=0\ , \tag{2.9}$$

$$\alpha_4\phi_u-3\alpha_6\phi_{uu}=0\ , \tag{2.10}$$

$$\alpha_5\phi_u+3\alpha_6\phi_{xu}=0\ . \tag{2.11}$$

Therefore, if system (2.8)-(2.11) is satisfied for coefficients (1.3) and function $\phi(t,x,u)$, the vc$K(m,n)$ (1.2) is nonlinearly self-adjoint. As an illustration of our nonlinear self-adjointness classification, we shall properly express coefficients (1.3) of vc$K(m,n)$ (1.2) in order to obtain some of its subclasses and their corresponding substitution function, $\phi(t,x,u)$.

2.1 Time-dependent KdV

Let us consider the following time-dependent KdV

$$u_t+\beta(t)uu_x+\gamma(t)u_{xxx}=0, \tag{2.12}$$

which is very useful for modeling positonic structures [40] and for describing the progression of weakly nonlinear and weakly dispersive waves in homogeneous media [1]. It can be derived from vc$K(m,n)$ (1.2) by defining coefficients (1.3) as

$$\alpha_0=0\ ,\quad \alpha_1=\beta(t)u\ ,\quad \alpha_2=\cdots=\alpha_5=0\ ,\quad \alpha_6=\gamma(t)\ . \tag{2.13}$$

From substitution of (2.13) into (2.8)-(2.11), we get

$$\phi(t,x,u)=k_1\left[u\int\beta(t)dt-x\right]+k_2u+k_3\ , \tag{2.14}$$

where k_1, k_2, and k_3 are arbitrary constants.

2.2 Cylindrical KdV

If coefficients (1.3) are written as

$$\alpha_0=0\ ,\quad \alpha_1=t^{-1/2}u\ ,\quad \alpha_2=\cdots=\alpha_5=0\ ,\quad \alpha_6=1\ , \tag{2.15}$$

and if additionally we consider $u(t,x) = t^{1/2}w(t,x)$, the vc$K(m,n)$ (1.2) becomes the cylindrical KdV equation

$$w_t + ww_x + w_{xxx} + \frac{1}{2t}w = 0 \tag{2.16}$$

which appears, for instance, in plasma physics [1]. By substituting coefficients (2.15) and the aforementioned transformation into system (2.8)-(2.11) results in

$$\phi(t,x,w) = k_1(2t^{3/2}w - t^{1/2}x) + k_2tw + k_3t^{1/2}, \tag{2.17}$$

wherein k_1, k_2, and k_3 are arbitrary constants.

2.3 Time-dependent mKdV

If coefficients (1.3) of vc$K(m,n)$ (1.2) are such that

$$\alpha_0 = 0\,, \quad \alpha_1 = 3fu^2\,, \quad \alpha_2 = \cdots = \alpha_5 = 0\,, \quad \alpha_6 = g, \tag{2.18}$$

and if the transformation $u(t,x) = \frac{1}{f}w(t,x)$, $f = f(t)$, is considered, we obtain the time-dependent mKdV

$$w_t - \frac{f'}{f}w + \frac{3}{f}w^2w_x + gw_{xxx} = 0 \tag{2.19}$$

that has appeared in different physical fields, including ocean dynamics, fluid mechanics, and plasma physics [45]. The substitution of coefficients (2.18) and the above mentioned transformation into system (2.8)-(2.11) leads to

$$\phi(t,x,w) = \frac{k_1}{f^2}w + \frac{k_2}{f}\,, \tag{2.20}$$

where k_1 and k_2 are arbitrary constants.

2.4 Time-dependent Schamel

The Schamel equation [36] $u_t + u^{1/2}u_x + \delta u_{xxx} = 0$, wherein δ is a constant, governs the propagation of ion-acoustic waves in a cold-ion plasma where some of the electrons do not behave isothermally during the passage of the wave but are trapped in it [28]. A time-dependent generalization of Schamel equation, i.e.,

$$u_t + A(t)u^{1/2}u_x + B(t)u_{xxx} = 0\,, \tag{2.21}$$

is obtained from vc$K(m,n)$ (1.2) if coefficients (1.3) are given by

$$\alpha_0 = 0\,, \quad \alpha_1 = A(t)u^{1/2}\,, \quad \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = 0\,, \quad \alpha_6 = B(t)\,. \tag{2.22}$$

It is important to emphasize that the time-dependent Schamel (2.21), which we have obtained as a subclass of vc$K(m,n)$ (1.2), does not correspond to the well-known time-dependent Schamel-KdV [2]. To the best of our knowledge, equation (2.21) with coefficients (2.22) was derived for the first time here. Provided that both convection and dispersion inhomogeneities of (2.21) are time-dependent, it might be potentially useful to describe highly nonlinear behavior of electrostatic structures in cold-ion plasmas. Substitution of coefficients (2.22) into system (2.8)-(2.11) leads to

$$\phi(t,x,u) = k_1 u + k_2 \,, \tag{2.23}$$

wherein k_1 and k_2 are arbitrary constants.

2.5 vc$K(m,n)$ (1.1)

Let us now consider the vc$K(m,n)$ (1.1),

$$u_t + [f(t,x)u^m + g(t,x)(u^n)_{xx}]_x = 0 \,,$$

where $f(t,x) = c_1 t x^{2m-7}/m$, $g(t,x) = c_2 t x^{2n-5}/n$, c_1 and c_2 non-zero constants. We substitute coefficients (1.3), with $f(t,x)$ and $g(t,x)$ of vc$K(m,n)$ (1.1), into system (2.8)-(2.11) to obtain

$$\phi(t,x,u) = A(t,x) + B(t)x^{-\left(\frac{2}{3}n-\frac{5}{3}\right)}u^n \,,$$

where $A(t,x)$ and $B(t)$ are related by the classifying equation

$$\begin{aligned} -27m\left[(4n^2-22n+30)A_x+(4n-10)xA_{xx}+x^2A_{xxx}\right]c_2tx^{2n-7}u^{n-1} \\ -27mA_tu^{n-1}-27B'mx^{-\frac{2}{3}n+\frac{5}{3}}u^{2n-1}-27c_1tmx^{2m-7}A_xu^{m-1} \\ +4m\left(8n^3-78n^2+249n-260\right)c_2tx^{\frac{4}{3}n-\frac{19}{3}}Bu^{2n-1} \\ +9\left[(8m-21)n-5m\right]c_1tx^{2m-\frac{2}{3}n-\frac{19}{3}}Bu^{m+n-1} = 0 \,. \end{aligned} \tag{2.24}$$

The equation (2.24) splits into two cases:

- $m \neq n$. We have $A(t,x) = c$ and $B(t) = 0$. Hence, $\phi(t,x,u) = c$, c constant.
- $m = n$. We find $B(t) = 0$, and then

$$\phi(t,x,u) = k_1 + k_2 x^{-2n+\frac{13}{2}+\frac{1}{2}\sqrt{1-\frac{4c_1}{c_2}}} + k_3 x^{-2n+\frac{13}{2}-\frac{1}{2}\sqrt{1-\frac{4c_1}{c_2}}} \,, \tag{2.25}$$

  k_1, k_2, and k_3 arbitrary constants. We assume that $(1-4c_1/c_2) > 0$ for convenience.

3 CONSERVATION LAWS BY USING A GENERAL THEOREM ON CONSERVATION LAWS

The following theorem was proved by Ibragimov [23, 24].

Theorem 1 (Ibragimov's theorem). *Let*

$$\mathrm{X} = \xi^i(x,u,u_{(1)},\ldots)\frac{\partial}{\partial x^i} + \eta(x,u,u_{(1)},\ldots)\frac{\partial}{\partial u}$$

be any Lie point, Lie-Bäcklund, or nonlocal symmetry of a given differential equation

$$F(x,u,u_{(1)},\ldots,u_{(s)}) = 0 \tag{3.1}$$

and

$$F^*(x,u,v,\ldots,u_{(s)},v_{(s)}) := \frac{\delta \mathscr{L}}{\delta u} = 0\,, \tag{3.2}$$

where $\mathscr{L} = vF$ is the formal Lagrangian, be the adjoint equation to equation (3.1). Then the combined system (3.1)-(3.2) has the conservation law $D_i(C^i) = 0$, where

$$\begin{aligned} C^i &= \xi^i \mathscr{L} + W\left[\frac{\partial \mathscr{L}}{\partial u_i} - D_j\left(\frac{\partial \mathscr{L}}{\partial u_{ij}}\right) + D_j D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}}\right) - \ldots\right] \\ &\quad + D_j(W)\left[\frac{\partial \mathscr{L}}{\partial u_{ij}} - D_k\left(\frac{\partial \mathscr{L}}{\partial u_{ijk}}\right) + \ldots\right] \\ &\quad + D_j D_k(W)\left[\frac{\partial \mathscr{L}}{\partial u_{ijk}} - \ldots\right] + \ldots\,, \end{aligned} \tag{3.3}$$

and $W = \eta - \xi^i u_i$.

In what follows, we employ this general theorem on conservation laws for some particular cases of scaling-invariant nonlinear dispersive subclasses of vc$K(m,n)$ (1.2) we have considered[1].

3.1 Time-dependent KdV

The time-dependent KdV (2.12), for $\beta(t) = \gamma(t) = t$, is written as

$$u_t + tuu_x + tu_{xxx} = 0\,, \tag{3.4}$$

whose formal Lagrangian is $\mathscr{L} = (u_t + tuu_x + tu_{xxx})\,v$.

Let us consider the scaling symmetry generator $\mathrm{X} = -4u\partial_u + 3t\partial_t + 2x\partial_x$. By adopting $x^1 = t, x^2 = x$ and substituing $v = -u$, the components (3.3) of the conserved vector are given by

$$C^1 = u^2\,, \qquad C^2 = 2tuu_{xx} - tu_x^2 + \frac{2}{3}tu^3\,. \tag{3.5}$$

We have obtained that the conserved functional corresponding to C^1 is the energy. $\int_{\mathfrak{R}} C^1\,dx$ is the integral of motion associated with the invariance under time shifts, i.e., $D_t\left[\int_{\mathfrak{R}} u^2\,dx\right] = 0$.

3.2 Cylindrical KdV

Let us now find a conservation law for the cylindrical KdV (2.16). Its formal Lagrangian assumes the form $\mathscr{L} = \left(w_t + ww_x + w_{xxx} + \frac{1}{2t}w\right)v$. By considering the scaling symmetry generator $\mathrm{X} = -2w\partial_w + 3t\partial_t + x\partial_x$ and substitution $v = 2t^{3/2}w - t^{1/2}x$ into (3.3), we find the conserved vector

$$C^1 = \frac{1}{2}t^{1/2}\left(tw^2 - xw\right)\,, \tag{3.6}$$

$$C^2 = \frac{1}{2}t^{1/2}\left(2tww_{xx} - xw_{xx} - tw_x^2 + w_x + \frac{2}{3}tw^3 - \frac{1}{2}xw^2\right)\,. \tag{3.7}$$

[1] The results presented in Sections 2 and 3 were obtained directly, by arduous calculations, but it is worth mentioning that symbolic computational packages are available to this end, such as, for instance, SYM [16] for Mathematica.

We should say that this is a particular case of a conservation law derived for the cylindrical KdV in [46] via multipliers method.

3.3 Time-dependent mKdV

The time-dependent mKdV (2.19), for $f = t^{-1}$ and $g = t$, is given by

$$w_t + t^{-1}w + 3tw^2 w_x + tw_{xxx} = 0 , \tag{3.8}$$

and its formal Lagrangian reads as $\mathscr{L} = \left(w_t + t^{-1}w + 3tw^2 w_x + tw_{xxx}\right) v$. By choosing the scaling symmetry generator $\mathrm{X} = -2w\partial_w + 3t\partial_t + 2x\partial_x$ and $v = t^2 u$, the corresponding conserved vector is

$$C^1 = t^2 w^2 , \qquad C^2 = \frac{3}{2}t^3 w^4 + 2t^3 ww_{xx} - t^3 w_x^2 . \tag{3.9}$$

3.4 Time-dependent Schamel

The time-dependent Schamel (2.21), for $A = B = t$, reads as

$$u_t + tu^{1/2}u_x + tu_{xxx} = 0 , \tag{3.10}$$

and its formal Lagrangian as $\mathscr{L} = \left(u_t + tu^{1/2}u_x + tu_{xxx}\right) v$. Let us consider the scaling symmetry generator $\mathrm{X} = u\partial_u - \frac{3}{8}t\partial_t - \frac{1}{4}x\partial_x$ and $v = u$. Therefore, the components of conserved vector of (3.10) are given by

$$C^1 = u^2 , \qquad C^2 = 2tuu_{xx} - 2tu_x^2 + \frac{4}{5}tu^{5/2} . \tag{3.11}$$

In this case, we have also obtained conservation of energy.

It is worth noting that a direct and relevant implication of conservation of $\int_{\Re} u^2\, dx$ is that if a solution $u(t,x)$ of either time-dependent KdV (3.4) or time-dependent Schamel (3.10) belongs to the space $L^2(\Re)$ at time $t = t_0$, then $u(t,x) \in L^2(\Re)$ for all $t \geq t_0$ [43].

4 CONSERVATION LAWS BY USING THE MULTIPLIERS METHOD

In 2003, Anco [3] showed within the multipliers method [5, 6] how to compute fluxes of conservation laws associated with scaling symmetries through a procedure that involves no integration. We follow reference [14] to briefly present this scaling-symmetry approach, restricting our notation to the case of partial differential equations with one dependent variable.

Consider a partial differential equation $F[u] = F(x,u,u_{(1)},\dots,u_{(s)}) = 0$, where s is the maximal order of derivatives, written in a solved form [2]. Suppose it is scaling-invariant under symmetry

$$\mathrm{X}[u] = p^{(i)}x^i \frac{\partial}{\partial x^i} + qu\frac{\partial}{\partial u} , \; i = 1,\dots,n \tag{4.1}$$

[2] An sth-order evolution equation $F(x,u,u_{(1)},\dots,u_{(s)}) = 0$ is written in a solved form for some leading derivative of u if all other terms in the equation contain neither the leading derivative nor its differential consequences [4]. The subclasses of vc$K(m,n)$(1.2) we consider are not only expressed in a solved form but also are equations of third-order Cauchy-Kovalevskaya form with respect to x.

where $p^{(i)}$ and q are called constant scaling weights of independent and dependent variables, respectively. From now on we adopt the notation "$f[u]$" for meaning a function of one or more independent variables x, a dependent variable u, and possibly derivatives of u up to some fixed order [14]. In the evolutionary form, the scaling symmetry generator (4.1) reads as $\hat{\mathrm{X}}[u] = \hat{\eta}[u]\partial_u$, wherein $\hat{\eta}[u] = qu - p^{(i)}x^i u_i$.

Let us assume $F[u]$ having a conservation law given by

$$\Lambda[u]F[u] = D_i C^i[u] \ , \tag{4.2}$$

wherein $\Lambda[u]$ are the multipliers of $F[u]$ and $C^i[u]$ the conservation law fluxes. Multipliers are obtained by solving the system of determining equations resulting from the variational derivative of (4.2), i.e.,

$$\frac{\delta}{\delta u}(\Lambda[u]F[u]) = 0 \ . \tag{4.3}$$

Hence, substitutions $\phi[u]$ of nonlinear self-adjointness condition (2.7) correspond to multipliers $\Lambda[u]$ derived from (4.3), and vice-versa.

Suppose now that $F[u]$ is homogeneous under the scaling symmetry (4.1), i.e.,

$$\mathrm{X}[u]F[u] = rF[u] \ , \tag{4.4}$$

where $r =$ constant is the scaling weight of $F[u]$. Assume that conservation law (4.2) is scaling-invariant and homogeneous under the scaling symmetry (4.1), i.e.,

$$\mathrm{X}[u]D_i C^i[u] = P D_i C^i[u] \ , \tag{4.5}$$

where P is a scaling weight of the conservation law. Then it is possible to show [8] that each multiplier $\Lambda[u]$ is homogeneous under scaling symmetry (4.1), i.e.,

$$\mathrm{X}[u]\Lambda[u] = s\Lambda[u] \ , \tag{4.6}$$

where $s = P - r$ is the scaling weight of each $\Lambda[u] \neq 0$. Therefore, if the following condition

$$\chi = s + r + \sum_{i=1}^{n} p^{(i)} \neq 0 \tag{4.7}$$

holds, the fluxes $C^i[u]$ of homogeneously scaling conservation law of scaling-invariant $F[u]$ can be computed through

$$C^i[u] = \sum_{p=0}^{s-1}\sum_{q=0}^{s-p-1} (-1)^q \left(D_{i_1}\dots D_{i_p}\hat{\eta}\right) D_{j_1}\dots D_{j_q}\left(\Lambda[u]\frac{\partial F[u]}{\partial u_{j_1\dots j_q i i_1\dots i_p}}\right) \ , \tag{4.8}$$

where s is the maximal order of derivatives appearing in $F[u]$, $j_1\dots j_q$ and $i_1\dots i_p$ are ordered combinations of indices such that $1 \leq j_1 \leq \dots \leq j_q \leq i \leq i_1 \leq \dots \leq i_p \leq n$, and n is the number of independent variables [3, 8, 14].

According to [14], for scaling-invariant $F[u]$ with scaling-homogeneous conservation law, this scaling-symmetry approach should be the preferred one since it demands the simplest computations of fluxes within the multipliers method. Therefore, we used the package GeM [14] for Maple to obtain fluxes of non-trivial conservation laws arising from local multipliers for particular cases of vc$K(m,n)$ (1.1), namely vc$K(2,2)$ and vc$K(3,3)$, which admit rational similarity (self-similar) solutions [38].

4.1 vc$K(2,2)$

The expanded form of this particular case of vc$K(m,n)$ (1.1) is given by

$$u_t + \alpha_0 + \alpha_1 u_x + \alpha_2 u_x^2 + \alpha_4 u_x u_{xx} + \alpha_5 u_{xx} + \alpha_6 u_{xxx} = 0\,, \tag{4.9}$$

where

$$\begin{aligned} \alpha_0 = -c_1 \frac{3t}{2x^4} u^2\,,\ \alpha_1 &= c_1 \frac{t}{x^3} u\,,\ \alpha_2 = -c_2 \frac{t}{x^2}\,, \\ \alpha_4 = c_2 \frac{3t}{x}\,,\ \alpha_5 &= -c_2 \frac{t}{x^2} u\,,\ \alpha_6 = c_2 \frac{t}{x} u\,. \end{aligned} \tag{4.10}$$

For vc$K(2,2)$ (4.9), regarding $c_1 = -2$ and $c_2 = 1$ in coefficients (4.10), we consider the scaling symmetry generator $\mathrm{X} = x\partial_x + 4u\partial_u$ and the following multipliers:

- $\Lambda[u] = x$. The corresponding density and flux are

$$C^1 = xu\,, \qquad C^2 = t\left(uu_{xx} + u_x^2\right) - \frac{t}{x} uu_x - \frac{3}{2}\frac{t}{x^2} u^2. \tag{4.11}$$

  This is a law of conservation of center of mass, i.e., $D_t\left[\int_{\Re} xu\, dx\right] = 0$.

- $\Lambda[u] = 1$. The density and flux obtained are

$$C^1 = u\,, \qquad C^2 = \frac{t}{x}\left(uu_{xx} + u_x^2\right) - \frac{t}{x^3} u^2\,. \tag{4.12}$$

  This result describes a law of conservation of mass, i.e., $D_t\left[\int_{\Re} u\, dx\right] = 0$.

4.2 vc$K(3,3)$

In its expanded form, this particular case of vc$K(m,n)$ (1.1) reads as

$$u_t + \alpha_0 + \alpha_1 u_x + \alpha_2 u_x^2 + \alpha_3 u_x^3 + \alpha_4 u_x u_{xx} + \alpha_5 u_{xx} + \alpha_6 u_{xxx} = 0\,, \tag{4.13}$$

where

$$\begin{aligned} \alpha_0 = -c_1 \frac{t}{3x^2} u^3\,,\ \alpha_1 &= c_1 \frac{t}{x} u^2\,,\ \alpha_2 = c_2 2tu\,,\ \alpha_3 = c_2 2tx\,, \\ \alpha_4 = c_2 6txu\,,\ \alpha_5 &= c_2 tu^2\,,\ \alpha_6 = c_2 txu^2\,. \end{aligned} \tag{4.14}$$

For vc$K(3,3)$ (4.13), regarding $c_1 = -6$ and $c_2 = 1$ in coefficients (4.14), we construct conservation laws associated to the scaling symmetry $\mathrm{X} = x\partial_x + u\partial_u$ and the following multipliers:

- $\Lambda[u] = x^3$. The density and flux obtained are

$$C^1 = x^3 u\,, \qquad C^2 = tx^4\left(u^2 u_{xx} + 2uu_x^2\right) - 3tx^3u^2u_x + tx^2u^3\,. \tag{4.15}$$

- $\Lambda[u] = 1$. The corresponding density and flux are

$$C^1 = u\,, \qquad C^2 = tx\left(u^2 u_{xx} + 2uu_x^2\right) - 2\frac{t}{x}u^3\,. \tag{4.16}$$

  In this case, we have also obtained a law of conservation of mass.

5 CONCLUDING REMARKS

This work brings together interesting features of nonlinear evolution equations, such as variable coefficients, scale invariance, and conserved quantities. An original nonlinear self-adjointness classification for a class of variable-coefficients nonlinear dispersive vc$K(m,n)$ was carried out. By means of a general theorem on conservation laws and the multipliers method, eight new local conservation laws associated with scaling symmetries for particular cases of scaling-invariant subclasses of vc$K(m,n)$ (1.2) were constructed. Among those eight original conservation laws, there are five with a direct physical interpretation: energy was the conserved quantity obtained for the particular time-dependent KdV (3.4) and time-dependent Schamel (3.10); for vc$K(2,2)$ (4.9), a law of conservation of center of mass and a law of conservation of mass were computed; for vc$K(3,3)$ (4.13), a law of conservation of mass was stablished. Additionally to the aforementioned results, it is worth noting that, to the best of our knowledge, the time-dependent Schamel (2.21), potentially useful to describe highly nonlinear behavior of electrostatic structures in cold-ion plasmas, was derived here for the first time as a subclass of vc$K(m,n)$ (1.2). In forthcoming studies, fractional vc$K(m,n)$ with noninteger m,n indices can be investigated.

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RESUMO. Simetrias de escala surgem em diferentes ramos da física e abordagens baseadas em simetria são poderosas ferramentas para estudar modelos invariantes por escala, pois podem fornecer leis de conservação que não são óbvias por inspeção. Nessa perspectiva, a classe de equações dispersivas não-lineares com coeficientes variáveis vc$K(m,n)$, que contém importantes equações de evolução que modelam fenômenos não-lineares, é considerada. Para algumas de suas subclasses invariantes por simetria de escala, estudamos sua auto-adjunticidade não-linear e construímos oito novas leis de conservação locais associadas a simetrias de escala, usando um teorema geral sobre leis de conservação e o método direto. A propriedade de invariância de escala dessas equações levou a cinco leis de conservação com uma interpretação física direta: energia, centro de massa e massa são as quantidades obtidas em alguns casos.

Palavras-chave: simetrias de escala, coeficientes variáveis, equações dispersivas não-lineares, auto-adjunticidade não-linear, leis de conservação.

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