Trends in Computational and Applied Mathematics

Trends in Computational and Applied Mathematics, **23**, N. 4 (2022), 673-682
Sociedade Brasileira de Matemática Aplicada e Computacional
Online version ISSN 2676-0029
www.scielo.br/tcam
doi: 10.5540/tcam.2022.023.04.00673

# Variational Formulation and *A Priori* Estimates for the Galerkin Method for a Fractional Diffusion Equation

M. E. de S. LIMA[1*], E. C. de OLIVEIRA[2] and A. da C. VIANA[3]

Received on December 23, 2021 / Accepted on May 20, 2022

**ABSTRACT.** In this work we obtain a variational formulation and *a priori* estimates for approximate solutions of a problem involving fractional diffusion equations.

**Keywords:** Galerkin method, fractional diffusion equation, *a priori* estimates.

## 1 INTRODUCTION

Fractional calculus has gained much prominence in recent decades, due to its applications in different fields of science, in particular, engineering, providing several useful tools to solve differential and integral equations and other problems involving special functions of mathematical physics, in addition to their extensions and generalizations in one and more variables. Among the various applications of fractional calculus we can cite the flow of a fluid, rheology, dynamic processes in self-similar and porous structures, diffusive transport similar to diffusion, electrical networks, probability and statistics, theory of control of dynamical systems and viscoelasticity (see [6]).

Anomalous diffusion can be characterized by both *L*evy flights, mathematically represented by the fractional Laplacian, as well as *l*ong rests, described by the time-fractional derivative. In this case, the appropriate equation, according to Schneider and Wyss [9] and Metzler and Klafter [8], is given by

$$u_t + D_t^{1-\alpha}(-\Delta)^{\gamma} u = 0, \tag{1.1}$$

*Corresponding author: Maria Elismara Sousa de Lima – E-mail: elismara_sousa@hotmail.com

[1]Instituto de Matemática, Estatística e Computação Científica, IMECC, Universidade Estadual de Campinas, R. Sérgio Buarque de Holanda, 651, 13083-859, Campinas, SP, Brasil – E-mail: elismara_sousa@hotmail.com https://orcid.org/0000-0001-6305-4936

[2]Departamento de Matemática Aplicada, IMECC, Universidade Estadual de Campinas, R. Sérgio Buarque de Holanda, 651, 13083-859, Campinas, SP, Brasil – E-mail: capelas@unicamp.br https://orcid.org/0000-0001-9661-0281

[3]Departamento de Matemática, UFS, Universidade Federal de Sergipe, Av. Marechal Rondon, s/n, 49100-000, São Cristóvão, SE,Brasil –E-mail: arlucioviana@academico.ufs.br https://orcid.org/0000-0003-2118-8279

where $\gamma \in (0,1)$ and $D_t^{\beta}\varphi$ denotes the fractional derivative of $\varphi$ of order $\beta > 0$ in the Riemann-Liouville sense, that is, $\alpha \in (0,1)$ (see Definition 2.2). In this way, the equation (1.1) can be rewritten as the equation

$$u_t + \partial_t \int_0^t g_\alpha(t-s)(-\Delta)^\gamma u(s)ds = 0, \tag{1.2}$$

where $g_\alpha$ is the function defined in (1.4).

Let us discuss the following problem for a fractional diffusion equation

$$\begin{cases} u_t + \partial_t(g_\alpha * (-\Delta)^\gamma u) = f, & \Omega \times [0,T], \\ u = 0, & \partial\Omega \times [0,T], \\ u(x,0) = u_0(x), & \Omega, \end{cases} \tag{1.3}$$

where $0 < \alpha < 1$, $T > 0$, $0 < \gamma < 1$, $\Omega$ is a smooth bounded domain of $\mathbb{R}^n$, $*$ denotes the convolution product and $g_\alpha$ is the Gel'fand Shilov function defined by

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \tag{1.4}$$

where $\Gamma$ is the Euler gamma function. The function $f$ belongs to $L^1(0,T;L^2)$ and also to $L^\infty(0,T;L^2)$. Furthermore, the fractional Laplacian operator can be defined in its spectral form by (see section 2.5.1 of [7]):

$$(-\Delta)^\gamma u(x) := \sum_{k=1}^{\infty} \lambda_k^\gamma (u, e_k)_{L^2(\Omega)} e_k(x), \tag{1.5}$$

where $\gamma \in (0,1)$, $\lambda_k$ are eigenvalues, and $e_k$ are eigenfunctions of $(-\Delta)$ with Dirichlet boundary conditions, that is,

$$\begin{aligned} -\Delta e_k &= \lambda_k e_k, \quad \text{in} \quad \Omega, \\ e_k &= 0, \quad \text{on} \quad \partial\Omega. \end{aligned}$$

Fractional-order diffusion equations describe anomalous diffusion phenomena, which help in the analysis of systems such as: plasma diffusion, fractal diffusion, anomalous diffusion on liquid surfaces, analysis of heart beat histograms in healthy individuals, among other physical systems (see [1] and [2]).

For the variational formulation of the problem we will use the integral form of Problem (1.3), given by

$$\begin{cases} u = u_0 - g_\alpha * (-\Delta)^\gamma u + 1 * f = 0, & \Omega \times [0,T], \\ u = 0, & \partial\Omega \times [0,T], \end{cases} \tag{1.6}$$

We will give the variational formulation and prove *a priori* estimates for the approximate solutions of the integral equation (1.6). Those results are useful to apply the Galerkin method (see [4]), which consists of finding approximate solutions to the problem, projecting it into finite-dimensional subspaces, dealing with fractional-order linear differential equations with initial values.

## 2 PRELIMINARIES

In this section we present some definitions and notations for the present work.

**Definition 2.1.** *Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval over $\mathbb{R}$. Riemann-Liouville fractional integrals, $I^{\alpha}_{a^+}$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ are given by:*

$$(I^{\alpha}_{a^+} f)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)ds,\ t > a,$$

*where $\Gamma(\alpha)$ is the gamma function and $f \in L^1[a,b]$.*

**Definition 2.2.** *Riemann-Liouville fractional derivatives, $D^{\alpha}_{a^+}$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ is given by*

$$(D^{\alpha}_{a^+} f)(t) = \left(\frac{d}{dt}\right)^n (I^{n-\alpha}_{a^+} f)(t),\ (n = [\alpha]+1, t > a),$$

*where $[\alpha]$ means the integer part of $\alpha$ and $f : I \to \mathbb{R}$. We take $n = \alpha$, if $\alpha \in \mathbb{N}_0$.*

**Definition 2.3.** *Caputo fractional derivative of order $\alpha$, on an interval $[a,b] \subset \mathbb{R}$, is given by*

$$({}^cD^{\alpha}_{a^+} \varphi)(t) := \left[ D^{\alpha}_{a^+} \left( \varphi(s) - \sum_{k=0}^{n-1} \frac{\varphi^{(k)}(a)}{k!} (s-a)^k \right) \right](t),$$

*where $n = [\alpha]+1$ if $\alpha \notin \mathbb{N}_0$ and $n = \alpha$, if $\alpha \in \mathbb{N}_0$.*

Note that the Problem (1.3) can be rewritten as

$$\begin{cases} u_t + D_t^{1-\alpha}(-\Delta_x)^{\gamma} u = f, & \Omega \times [0,T], \\ \\ u(x,0) = u_0(x), & \Omega. \end{cases} \tag{2.1}$$

Where $0 < \gamma < 1$ and $0 < \alpha < 1$. In fact, since $0 < \alpha < 1$ and $1-\alpha < 1$, we have to

$$D_t^{1-\alpha}[(-\Delta)^{\gamma} u] = \partial_t \left( g_{\alpha} * (-\Delta)^{\gamma} u \right).$$

We will use the following spaces $L^{\infty}(0,T;L^2(\Omega))$, $L^2(0,T;H^{\gamma}(\Omega))$ and $L^1(0,T;L^2(\Omega))$, where $\Omega$ is an open on $\mathbb{R}^n$. We remember that $L^p(\Omega)$ is the space of all measurable functions $f : \Omega \to \mathbb{R}$, with $\|f\|_{L^p(\Omega)} < \infty$ such that

$$\|f\|_{L^p(\Omega)} := \begin{cases} \left(\int_{\Omega} |f|^p dx\right)^{1/p}, & \text{if} \quad 1 \le p < \infty, \\ \operatorname{ess\,sup}_{\Omega} |f|, & \text{if} \quad p = \infty. \end{cases} \tag{2.2}$$

**Definition 2.4.** *Let $X$ a Banach space. The space $L^p(0,T;X)$ consists of all measurable functions*

$$u : (0,T) \to X$$

*with*

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{1/p} < \infty$$

*for* $1 \leq p < \infty$, *and*

$$\|u\|_{L^\infty(0,T;X)} = \sup_{t\in(0,T)} \|u(t)\|_X < \infty.$$

For simplicity, we sometimes denote $L^p(0,T;L^p(\Omega))$ by $L^p(0,T;L^p)$. Furthermore, we denote the inner product in $L^2$ by $(\cdot,\cdot)$ and in $H^\gamma$ by $(\cdot,\cdot)_{H^\gamma}$.

The fractional Sobolev space $H^\gamma$ is a Hilbert space and is defined below.

**Definition 2.5 (Definition A.5, [7]).** *For any* $\gamma \geq 0$

$$H^\gamma(\Omega) := \left\{ u = \sum_{k=1}^{\infty} u_k e_k \in L^2(\Omega) : \|u\|^2_{H^\gamma(\Omega)} := \sum_{k=1}^{\infty} \lambda_k^\gamma u_k^2 < \infty \right\}, \tag{2.3}$$

*where* $(\lambda_k, e_k)$ *are the eigenvalues and their respective eigenvectors of* $(-\Delta)$ *with Dirichlet boundary conditions, whose norm coincides with* $\|(-\Delta)^{\gamma/2}u\|_{L^2}$, *according to equation* (1.5).

Before we present Theorem 2.1 we need the following definitions.

**Definition 2.6.** *Let* $A \in M_n(\mathbb{R})$, $z \in \mathbb{C}$ *and* $\alpha > 0$. *We define the matrix* $\alpha$*-exponential function by*

$$e_\alpha^{Az} := z^{\alpha-1} \sum_{k=0}^{\infty} A^k \frac{z^{\alpha k}}{\Gamma[(k+1)\alpha]}.$$

**Definition 2.7.** *A weighted space of continuous functions is of the form*

$$C_{n-\alpha}[a,b] = \left\{ g(t) : (t-a)^{n-\alpha} g(t) \in C[a,b], \|g\|_{Cn-\alpha} = \left\|(t-a)^{n-\alpha} g(t)\right\|_C \right\}.$$

We use the following existence and uniqueness theorem for a Cauchy problem of a fractional matrix equation with a Caputo derivative (see [6]).

**Theorem 2.1 (Theorem 7.14, [6]).** *The following initial value problem*

$$({}^cD^\alpha_{a^+}\overline{Y})(x) = A\overline{Y}(x) + \overline{B}(x), \tag{2.4}$$

$$\overline{Y}(a) = \overline{b}, \, (\overline{b} \in \mathbb{R}^n), \tag{2.5}$$

*where* $A \in M_n(\mathbb{R})$ *and* $\overline{B} \in \overline{C}_{1-\alpha}([a,b])$, *has a single continuous solution given by*

$$\overline{Y} = \int_a^x e_\alpha^A(x-\xi)[\overline{B}(\xi) + A\overline{b}]\, d\xi + \overline{b}. \tag{2.6}$$

Also, we need the following result which can be found in [3] and references therein.

**Theorem 2.2.** *Let $(H,(\cdot,\cdot))$ be a real Hilbert space, $f \in L^2(0,T;H)$ and $\alpha \in (0,1)$. Then*

$$\int_0^T (f(t), g_\alpha * f(t))\,dt \geq 0. \tag{2.7}$$

**Lemma 2.1 (Lemma 2.22, [6]).** *Let $\alpha > 0$ and let n be given by $n = [\alpha]+1$, if $\alpha \notin \mathbb{N}$ and $n = \alpha$, if $\alpha \in \mathbb{N}_0$. If $y(x) \in AC^n[a,b]$ or $y(x) \in C^n[a,b]$, then*

$$\left(I_{a^+}^{\alpha}\ {}^{c}D_{a^+}^{\alpha} y\right)(x) = y(x) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!}(x-a)^k \tag{2.8}$$

*In particular, if $0 < \alpha < 1$ and $y(x) \in AC[a,b]$ or $y(x) \in C[a,b]$, then*

$$\left(I_{a^+}^{\alpha}\ {}^{c}D_{a^+}^{\alpha} y\right)(x) = y(x) - y(a). \tag{2.9}$$

**Definition 2.8.** *We define the Sobolev space*

$$H^1_{1,loc}(\mathbb{R}_+) := \{\varphi \in L^1_{loc}(\mathbb{R}_+); \varphi' \in L^1_{loc}(\mathbb{R}_+)\}. \tag{2.10}$$

**Lemma 2.2 (Lemma 6.2, [5]).** *Let $T > 0$ and $\Omega \subset \mathbb{R}^d$ be an open set. Let $k \in H^1_{1,loc}(\mathbb{R}_+)$ be nonnegative and nonincreasing function. Then for any $v \in L^2((0,T)\times\Omega)$ and any $v_0 \in L^2(\Omega)$ there holds*

$$\int_\Omega v\partial_t(k*[v-v_0])dx \geq |v(t)|_{L^2(\Omega)}\partial_t\left(k*\left[|v|_{L^2(\Omega)} - |v_0|_{L^2(\Omega)}\right]\right)(t),$$

*for each $t \in (0,T)$.*

## 3 MAIN RESULTS

In this section, we obtain the variational formulation and *a priori* estimates for the approximate solutions of Problem (1.3). Here, we perform formal calculations so that $u$ is considered as regular as necessary.

### 3.1 Variational formulation

For the variational formulation we will use the integral form of Problem (1.3), given by

$$u = u_0 - g_\alpha * (-\Delta)^\gamma u + 1 * f, \tag{3.1}$$

as long as the fractional Laplacian applied to $u$ is continuous, where $u = 0$ on the boundary of $\Omega$ and $\gamma \in (0,1)$. Multiplying (3.1) by $v \in H^\gamma$ such that $v = v(x)$ and, integrating over $\Omega$, we have

$$\int_\Omega uv\,dx = \int_\Omega u_0 v\,dx - \int_\Omega (g_\alpha * (-\Delta)^\gamma u)v\,dx + \int_\Omega (1*f)v\,dx. \tag{3.2}$$

Thus, using Fubini theorem and knowing that the fractional Laplacian is self-adjoint on $L^2$, in addition to having the semigroup property, we obtain

$$\begin{aligned}
\int_\Omega (g_\alpha * (-\Delta)^\gamma u) v\,dx &= \int_0^t g_\alpha(t-s) \int_\Omega (-\Delta)^\gamma u(s,x) v(x)\,dx\,ds \\
&= \int_0^t g_\alpha(t-s) \int_\Omega (-\Delta)^{\frac{\gamma}{2}} u(s,x) (-\Delta)^{\frac{\gamma}{2}} v(x)\,dx\,ds \\
&= \int_\Omega \left( \int_0^t g_\alpha(t-s)(-\Delta)^{\frac{\gamma}{2}} u(s,x)\,ds \right) (-\Delta)^{\frac{\gamma}{2}} v(x)\,dx \\
&= (g_\alpha * (-\Delta)^{\frac{\gamma}{2}} u, (-\Delta)^{\frac{\gamma}{2}} v).
\end{aligned}$$

Thus, it follows from equation (3.2) that

$$(\boldsymbol{u}, v) = (u_0, v) - (g_\alpha * (-\Delta)^{\frac{\gamma}{2}} u, (-\Delta)^{\frac{\gamma}{2}} v) + (1 * f, v) \tag{3.3}$$

or

$$(\boldsymbol{u}, v) = (u_0, v) - (g_\alpha * u, v)_{H^\gamma} + (1 * f, v), \tag{3.4}$$

where the equation (3.4) gives us the variational form of the problem. We denote $(g_\alpha * u, v)_{H^\gamma}$ by $B[u, v; t]$. Now, let us build the approximate solutions. For this, consider a base $\{v_k\}_k$ orthogonal to $H^\gamma$ that is orthonormal to $L^2(\Omega)$.

For every natural number $m$, consider the vector subspace

$$V^m = [v_1, \cdots, v_m]$$

and,

$$u_m(t) = \sum_{j=1}^m \beta_m^j(t) v_j, \tag{3.5}$$

where we must determine the coefficients $\beta_m^j(t)$ $(0 \le t \le T$ and $j = 1, \cdots, m)$ such that

$$\beta_m^j(0) = (u_0, v_j) \qquad j = 1, \cdots, m \tag{3.6}$$

and

$$(u_m, v_j) = \beta_m^j(0) - B[u_m, v_j; t] + (1 * f, v_j)\,. \tag{3.7}$$

**Theorem 3.3.** *If $f \in L^\infty(0,T;L^2)$, then for every integer $m = 1, 2, \cdots$, there is a single differentiable function $u_m$, given by* (3.5)*, satisfying equations* (3.6) *and* (3.7).

**Proof.** Suppose $u_m$ has the form equation (3.5). The proof consists in to show the existence and uniqueness of $\beta_m^j(t)$. So,

$$(u_m, v_k) = \left( \sum_{j=1}^m \beta_m^j(t) v_j, v_k \right) = \beta_m^k(t),$$

because $\{v_j\}_j$ is orthonomal. Furthermore,

$$\begin{aligned} B[u_m, v_k; t] &= (g_\alpha * (-\Delta)^{\frac{\gamma}{2}} u_m, (-\Delta)^{\frac{\gamma}{2}} v_k) = g_\alpha * ((-\Delta)^{\frac{\gamma}{2}} \sum_{j=1}^{m} \beta_m^j(t) v_j, (-\Delta)^{\frac{\gamma}{2}} v_k) \\ &= g_\alpha * (\sum_{j=1}^{m} \beta_m^j(t)(-\Delta)^{\frac{\gamma}{2}} v_j, (-\Delta)^{\frac{\gamma}{2}} v_k) = \sum_{j=1}^{m} (g_\alpha * \beta_m^j)(t)(v_j, v_k)_{H^\gamma} \\ &= \sum_{j=1}^{m} (g_\alpha * \beta_m^j)(t) e^{jk}, \end{aligned}$$

where $e^{jk} = (v_j, v_k)_{H^\gamma}$. Define $f^k(t) = (1 * f(t), v_k)$. So, from equation (3.7), we have

$$\beta_m^k(t) - \beta_m^j(0) + \sum_{j=1}^{m} e^{jk}(g_\alpha * \beta_m^j)(t) = f^k(t). \tag{3.8}$$

Let $X = \begin{bmatrix} \beta_m^1(t) \\ \vdots \\ \beta_m^n(t) \end{bmatrix}$, $X^0 = \begin{bmatrix} \beta_m^1(0) \\ \vdots \\ \beta_m^n(0) \end{bmatrix}$, $A = [e^{ij}]$ and $F = \begin{bmatrix} (f, v_1) \\ \vdots \\ (f, v_m) \end{bmatrix}$.

We can rewrite (3.8) in the following matrix form

$$X - X^0 + g_\alpha * (AX) = 1 * F, \tag{3.9}$$

So, equation (3.9) can be rewritten as

$$g_{1-\alpha} * (X - X^0) + 1 * (AX) = 1 * g_{1-\alpha} * F \Rightarrow^c D^\alpha X + AX = g_{1-\alpha} * F\,.$$

Thus, by hypothesis, as $f \in L^\infty(0,T; L^2)$, it follows that $g_{1-\alpha} * F \in C_{1-\alpha}([0,T])$. Therefore, by Theorem 2.1, it follows the existence and uniqueness of $\beta_m^j$. $\square$

### 3.2 *A priori* estimates

In this section we prove *a priori* estimates given by the following theorem.

**Theorem 3.4.** *Let $\alpha \in (0,1)$. If $f \in L^1(0,T; L^2)$, then*

$$\|u_m\|_{L^\infty(0,T;L^2)} \leq \|u_{0m}\|_{L^2} + \|f\|_{L^1(0,T;L^2)}. \tag{3.10}$$

*If, additionally, $f \in L^\infty(0,T; L^2)$, then*

$$\|u_m\|_{L^1(0,T;H^\gamma)} \leq \frac{T^{1-\alpha}}{c\Gamma(2-\alpha)} \|u_{0m}\|_{L^2} + \frac{T^{\frac{3-\alpha}{2}}}{c\Gamma(2-\alpha)^{\frac{1}{2}}} \|f\|_{L^\infty(0,T;L^2)}. \tag{3.11}$$

**Proof.** Since $u_m$ is the function defined in (3.5) and guaranteed by Theorem 3.3, we multiply equation (3.7) by $\beta_m^j$ and sum with $j$ running from 1 to $m$, to get

$$\|u_m\|_{L^2}^2 = (u_{0m}, u_m) - (g_\alpha * u_m, u_m)_{H^\gamma} + (1 * f, u_m)_{L^2}. \tag{3.12}$$

We note that $u_m \in L^2(0,T;H^\gamma)$. In fact, looking at the expression (3.5) we can infer that

$$\begin{aligned}\|u_m(t)\|^2_{L^2(0,T;H^\gamma)} &\leq \sum_{j=1}^m \int_0^T \|\beta_m^j(t)v_j(\cdot)\|^2_{H^\gamma}dt \leq \sum_{j=1}^m \int_0^T |\beta_m^j(t)|^2\,dt\|v_j\|_{H^\gamma}\\ &= \sum_{j=1}^m \|\beta_m^j(t)\|^2_{L^2(0,T)}\|v_j\|_{H^\gamma} < \infty,\end{aligned}$$

since $\beta_m^j \in L^2(0,T)$, $v_j \in H^\gamma$ and the sum is finite. So, by Theorem 2.2, we have

$$(g_\alpha * u_m, u_m)_{H^\gamma} \geq 0.$$

It follows from this and from equation (3.12) that

$$\|u_m\|^2_{L^2} \leq (u_{0m}, u_m)_{L^2} + (1*f, u_m)_{L^2} \leq \|u_{0m}\|_{L^2}\|u_m\|_{L^2} + \|f\|_{L^1(0,T;L^2)}\|u_m\|_{L^2},$$

where we have used Hölder inequality and Minkowski integral inequality. Hence,

$$\|u_m(t)\|_{L^2} \leq \|u_{0m}\|_{L^2} + \|f\|_{L^1(0,T;L^2)}. \tag{3.13}$$

This proves (3.10). For the proof of estimate (3.11) let us observe that

$$u = u_0 - g_\alpha * (-\Delta)^\gamma u + 1 * f \Rightarrow^c D_t^\alpha u = -(-\Delta)^\gamma u + g_{1-\alpha} * f, \tag{3.14}$$

so that,

$$({}^cD_t^\alpha u, u) + \|u\|^2_{H^\gamma} = (g_{1-\alpha} * f, u). \tag{3.15}$$

But, putting $k = g_{1-\alpha}$ in Lemma 2.2, we have

$$({}^cD_t^\alpha u, u) \geq \|u\|_{L^2}\ {}^cD_t^\alpha \|u\|_{L^2}, \tag{3.16}$$

with $u \in L^2(0,T)$.

By estimating (3.10), it is immediate to see that $u_m \in L^2(0,T;L^2)$. So (3.16) holds for $u_m$. Therefore, (3.15) gives us

$$\|u_m\|_{L^2}\ {}^cD_t^\alpha \|u_m\|_{L^2} + \|u_m\|^2_{H^\gamma} \leq \|g_{1-\alpha} * f\|_{L^2}\|u_m\|_{L^2}. \tag{3.17}$$

From the continuous inclusion $L^2 \hookleftarrow H^\gamma$, it follows that there is a constant $c > 0$ such that $c\|u_m\|_{L^2} \leq \|u_m\|_{H^\gamma}$. Therefore,

$$c\|u_m\|_{L^2}\left[\frac{1}{c}\ {}^cD_t^\alpha \|u_m\|_{L^2} + \|u_m\|_{H^\gamma}\right] \leq \|g_{1-\alpha} * f\|_{L^2}\|u_m\|_{L^2}, \tag{3.18}$$

implying

$${}^cD_t^\alpha \|u_m\|_{L^2} + c\|u_m\|_{H^\gamma} \leq \|g_{1-\alpha} * f\|_{L^2}. \tag{3.19}$$

Applying $I^{\alpha}_{0^+}$ to (3.19), we have by Lemma 2.1

$$\|u_m\|_{L^2} - \|u_{0m}\|_{L^2} + cg_\alpha * \|u_m\|_{H^\gamma} \le g_\alpha * \|g_{1-\alpha} * f\|_{L^2}, \tag{3.20}$$

implying

$$g_\alpha * \|u_m\|_{H^\gamma} \le \frac{1}{c}\left[\|u_{0m}\|_{L^2} + g_\alpha * \|g_{1-\alpha} * f\|_{L^2}\right]. \tag{3.21}$$

Using Minkowski integral inequality, we can estimate $\|g_{1-\alpha} * f\|_{L^2}$. In fact,

$$\begin{aligned}
\|g_{1-\alpha} * f\|^2_{L^2} = &\int_\Omega \left|\int_0^t g_{1-\alpha}(t-s) f(s,x) ds\right|^2 dx \\
\le &\int_\Omega \int_0^t |f(s,x)|^2 dg_{2-\alpha}(t-s) dx \\
\le &\int_0^t g_{1-\alpha}(t-s) ds \|f\|^2_{L^\infty(0,T;L^2)} \\
= &\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} \|f\|^2_{L^\infty(0,T;L^2)} \\
\le &\frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|f\|^2_{L^\infty(0,T;L^2)},
\end{aligned}$$

for $t \in (0,T)$. Then, we can write

$$\|g_{1-\alpha} * f\|_{L^2} \le \frac{T^{(1-\alpha)/2}}{\Gamma(2-\alpha)^{1/2}} \|f\|_{L^\infty(0,T;L^2)}. \tag{3.22}$$

So, applying $g_{1-\alpha}$ to (3.21), we have

$$\|u_m\|_{L^1(0,T;H^\gamma)} \le \frac{T^{1-\alpha}}{c\Gamma(2-\alpha)} \|u_{0m}\|_{L^2} + \frac{T^{\frac{3-\alpha}{2}}}{c\Gamma(2-\alpha)^{\frac{1}{2}}} \|f\|_{L^\infty(0,T;L^2)}, \tag{3.23}$$

which is the desired result. □

## 4 CONCLUDING REMARKS

In this work we obtained the variational formulation and an estimate *a priori* of Problem (2.1), results that will help us to apply Galerkin's method and enable us to prove the existence and uniqueness of the solution to Problem (2.1). Later, we will investigate the existence of global solutions and their stability. Also, one will be able to implement numerical simulations.

## REFERENCES


[1] P. Biler, T. Funaki & W.A. Woyczynski. Fractal Burgers equations. *Journal of Differential Equations*, **148**(1) (1998), 9–46.

[2] M.P. Bonkile, A. Awasthi, C. Lakshmi, V. Mukundan & V. Aswin. A systematic literature review of Burgers' equation with recent advances. *Pramana*, **90**(6) (2018), 1–21.

[3] L. Djilali & A. Rougirel. Galerkin method for time fractional diffusion equations. *Journal of Elliptic and Parabolic Equations*, **4**(2) (2018), 349–368.

[4] L.C. Evans. Partial Differential Equations. *Graduate Studies in Mathematics*, **19**(4) (1998), 7.

[5] J. Kemppainen, J. Siljander, V. Vergara & R. Zacher. Decay estimates for time-fractional and other non-local in time subdiffusion equations in $\mathbb{R}^d$. *Mathematische Annalen*, **366**(3) (2016), 941–979.

[6] A.A. Kilbas, H.M. Srivastava & J.J. Trujillo. "Theory and Applications of Fractional Differential Equations", volume 204. Elsevier (2006).

[7] A. Lischke, G. Pang, M. Gulian, F. Song, C. Glusa, X. Zheng, Z. Mao, W. Cai, M.M. Meerschaert, M. Ainsworth *et al.* What is the fractional Laplacian? A comparative review with new results. *Journal of Computational Physics*, **404** (2020), 109009.

[8] R. Metzler & J. Klafter. The random walk's guide to anomalous diffusion: a fractional dynamics approach. *Physics Reports*, **339**(1) (2000), 1–77.

[9] W.R. Schneider & W. Wyss. Fractional diffusion and wave equations. *Journal of Mathematical Physics*, **30**(1) (1989), 134–144.