Trends in Computational and Applied Mathematics

Trends in Computational and Applied Mathematics, **24**, N. 1 (2023), 177-190
Sociedade Brasileira de Matemática Aplicada e Computacional
Online version ISSN 2676-0029
www.scielo.br/tcam
doi: 10.5540/tcam.2022.024.01.00177

Sufficient Conditions for Existence of the LU Factorization of Toeplitz Symmetric Tridiagonal Matrices

C. G. ALMEIDA[1*] and S. A. E. REMIGIO[2]

Received on March 21, 2022 / Accepted on August 8, 2022

ABSTRACT. The characterization of inverses of symmetric tridiagonal and block tridiagonal matrices and the development of algorithms for finding the inverse of any general non-singular tridiagonal matrix are subjects that have been studied by many authors. The results of these research usually depend on the existence of the LU factorization of a non-sigular matrix A, such that $A = LU$. Besides, the conditions that ensure the nonsingularity of A and its LU factorization are not promptly obtained. Then, we are going to present in this work two extremely simple sufficient conditions for existence of the LU factorization of a Toeplitz symmetric tridiagonal matrix A. We take into consideration the roots of the modified Chebyshev polynomial, and we also present an analysis based on the parameters of the Crout's method.

Keywords: Toeplitz tridiagonal matrix, Crout's method, tridiagonal and diagonally dominant matrix.

1 INTRODUCTION

The development of algorithms for finding the inverse of any general non-singular tridiagonal or pentadiagonal matrix, [9], [20], [14], and [1] (see also the references in these papers), and the characterization of inverses of symmetric tridiagonal and block tridiagonal matrices are subjects that have been studied by many authors. Meurant's paper [16], from 1992, presents a good review on these research. According to this author, closed form explicit formulas for elements of the inverses can only be given for special matrices, e.g., Toeplitz tridiagonal matrices [10] corresponding, for instance, to constant coefficients 1D elliptic partial differential equations (pde), or for block matrices arising from separable 2D elliptic pde [2].

The results of these research usually depend on the existence of the LU factorization of a non-sigular matrix A, such that $A = LU$. Besides, in the most of the papers, or it is assumed that the matrix is invertible [2, 6, 10, 16] or the conditions that ensure the nonsingularity of A and its LU factorization are not promptly obtained [7, 8].

*Corresponding author: César Guilherme Almeida – E-mail: cesargui@ufu.br

[1]Núcleo de Matemática Aplicada da Faculdade de Matemática, Universidade Federal de Uberlândia, 38408-100, Brazil – E-mail: cesargui@ufu.br https://orcid.org/0000-0001-7619-8162

[2]Núcleo de Matemática Aplicada da Faculdade de Matemática, Universidade Federal de Uberlândia, 38408-100, Brazil – E-mail: santos.er@ufu.br https://orcid.org/0000-0002-5088-7311

For example, in Meurant's paper [16] some results concerning the characterization of inverses of symmetric tridiagonal and block tridiagonal matrices were obtained by relating the elements of inverses to elements of the Cholesky decompositions of these matrices. Elmikkawy in his paper from 2002 [8] presented conditions for a symmetric tridiagonal matrix to be positive definite and to have a Cholesky decomposition. These conditions were based on the parameters of the Crout's method.

Recent research continues to highlight the importance of studying Topelitz matrices, such as [3, 12, 13, 19].

Yaru Fu et al. [12], in 2020, presented in their paper some properties for a class of perturbed Toeplitz periodic tridiagonal (PTPT) matrices, including the determinant, and the inverse matrix. Specifically, the determinant of the PTPT matrix can be explicitly expressed using the well-known Fibonacci numbers. This technique is different from that used in our work.

Another technique which is also different from that used in our work was presented in the paper of Yunlan Wei et al. [19], in 2019. In that paper, the authors derived the formulas on representation of the determinants and inverses of the periodic tridiagonal Toeplitz matrices with perturbed corners of type I in the form of products of Fermat numbers and some initial values.

Zhongyun Liu et al. [13], in 2020, developed in their paper fast solvers for tridiagonal Toeplitz linear systems. However, the authors did not present sufficient conditions for existence of the LU factorization of Toeplitz tridiagonal matrices. In 2021, Skander Belhaj et al. [3] also developed in their paper a fast algorithm for solving diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems. Numerical experiments were given in order to illustrate the validity and efciency of the algorithm. However, in the same way as before, the authors did not present sufficient conditions for existence of the LU factorization of Toeplitz matrices.

In our work we are going to consider two extremely simple sufficient conditions for existence of the LU factorization of a Toeplitz symmetric tridiagonal matrix A. We will show that if $0 < |d| < 2|a|$, $|d| \neq |a|$, and $|d|/|a|$ is a rational number, then A has an LU decomposition and $\det(A) \neq 0$, where d is the element that belongs to the main diagonal of A, and a is the element that belongs to the first diagonal above the main diagonal. Besides, we will show that if $|d| \geq 2|a| > 0$, then A is non-singular and has an LU decomposition. This last result is a consequence of the theorem (presented in our work) that considers a tridiagonal diagonally dominant matrix A (not strictly diagonally dominant matrix).

The condition presented in the first above case extends the work of Fischer and Usmani [10] that had only considered $-d/a > 0$, and not presented conditions assuring that $\det(A) \neq 0$, when $0 < -d/a < 2$. We take into consideration, in our work, the analysis of the roots of the same modified Chebyshev polynomial that was used in the Bank's paper [2]. With respect to the second condition, we have also presented an analysis based on the parameters of the Crout's method. We considered a tridiagonal diagonally dominant matrix A (not strictly diagonally dominant matrix) and obtained a very simple criterion for detecting when such matrix A is non-singular and has an LU decomposition.

There are multiple studies involving tridiagonal matrices and, specially, diagonally dominant matrices. For instance, Peter Z. Revesz, in his article [18], "Cubic spline interpolation by solving a recurrence equation instead of a tridiagonal matrix", described a method that can be used in a wide variety of applications which require interpolation of a function of one variable. In his words, for example, interpolation of measurement data can generate constraint databases that can be efficiently queried using constraint query languages (see reference [17]).

According to McNally [14], "Banded Toeplitz systems of linear equations arise in many application areas and have been well studied in the past. Recently, significant advancement has been made in algorithm development of fast parallel scalable methods to solve tridiagonal Toeplitz problems". That paper presented a new algorithm for solving symmetric pentadiagonal Toeplitz systems of linear equations based on a technique used in [15] for tridiagonal Toeplitz systems. "A common example which arises in natural quintic spline problems has been used to demonstrate the algorithm's effectiveness".

We have developed a theoretical study, tanking into consideration the previous presentation, that culminated in a low-cost test for detecting in a simple way when a Toeplitz symmetric tridiagonal matrix is non-singular and has an LU decomposition.

The test is introduced in Theorem 3.3 from Section 3. One part of this test is based on Crout's method (see Equation (2.2)) and uses a criterion that is presented in Theorem 2.1 from Section 2. The other part of the test uses a criterion based on calculations of the principal minors from a Toeplitz symmetric tridiagonal matrix, and theory of polynomials.

Finally, the paper is organized as follows:

- Section 2: some definitions will be presented as well as preliminary results for tridiagonal matrices.
- Section 3: we will show, in this section, preliminary results for symmetic tridiagonal Toeplitz matrices and we will prove the main result of our work.
- Section 4: this final section will present the conclusions of the work.

2 DEFINITIONS AND PRELIMINARY RESULTS FOR TRIDIAGONAL MATRICES

In this work, a tridiagonal matrix A, with real elements, will be given by:

$$A = \begin{bmatrix} d_1 & a_1 & 0 & & & \dots & 0 \\ b_2 & d_2 & a_2 & & & & \vdots \\ 0 & b_3 & d_3 & a_3 & & & \\ & & \ddots & \ddots & \ddots & & \\ \vdots & & & & & & \\ & & & & b_{n-1} & d_{n-1} & a_{n-1} \\ 0 & \dots & & & & b_n & d_n \end{bmatrix}. \tag{2.1}$$

In this case, we consider $b_1 = 0 = a_n$.

We will say that A is diagonally dominant matrix if, and only if, for all i, $1 \le i \le n$, $|d_i| \ge |b_i| + |a_i|$. Besides, if $|d_i| > |b_i| + |a_i|$, for all i, $1 \le i \le n$, then A is a strictly diagonally dominant matrix.

There is a way to prove a matrix A has LU decomposition wich consists of demonstrating that its principal minors are not null (see, for example, [11]). The principal minor of order m from a matrix A of order n, $1 \le m \le n$, is the determinant of the submatrix composed by the first m rows and m columns of the matrix A.

Remark: An important result (see, for example, [4]) states that every strictly diagonally dominant matrix is non-singular and has LU decomposition.

Let A be a tridiagonal matrix as shown in Equation (2.1). According to [9,20], we know that if $A = LU$, then L and U are tridiagonal matrices given by:

$$L = \begin{bmatrix} \alpha_1 & 0 & 0 & & \dots & 0 \\ b_2 & \alpha_2 & 0 & & & \vdots \\ 0 & b_3 & \alpha_3 & & & \\ & & \ddots & \ddots & \ddots & \\ \vdots & & & b_{n-1} & \alpha_{n-1} & 0 \\ 0 & \dots & & & b_n & \alpha_n \end{bmatrix}, U = \begin{bmatrix} 1 & \frac{a_1}{\alpha_1} & 0 & & \dots & 0 \\ 0 & 1 & \frac{a_2}{\alpha_2} & & & \vdots \\ & 0 & 1 & \frac{a_3}{\alpha_3} & & \\ & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & 0 & 1 & \frac{a_{n-1}}{\alpha_{n-1}} \\ 0 & \dots & & & 0 & 1 \end{bmatrix}, \tag{2.2}$$

where $\alpha_1 = d_1$, $\gamma_1 = \dfrac{a_1}{\alpha_1}$ and $\alpha_i = d_i - b_i\gamma_{i-1} = d_i - b_i\dfrac{a_{i-1}}{\alpha_{i-1}}$, $2 \le i \le n$.

The previous decomposition (2.2), considering $U_{ii} = 1$, $1 \le i \le n$, is known as Crout's decomposition (see [9,20]). This decomposition is always possible whenever $\alpha_i \ne 0$, $1 \le i \le n$. In this case, we obtain that $\det(A) \ne 0$. The first theorem below presents a case where $\alpha_i \ne 0$, $1 \le i \le n$.

Theorem 2.1. *Let A be a tridiagonal diagonally dominant matrix as shown in Equation (2.1). Suppose there is an integer k, $1 < k \le n$, such that $\alpha_i \ne 0$ and $|\gamma_i| \le 1$, $1 \le i \le k-1$, $|d_{k-1}| > |b_{k-1}| + |a_{k-1}|$, $|d_i| \ge |b_i| + |a_i|$, $i \in \{k, \dots, n\}$, and $b_{k+j} \ne 0$, $0 \le j \le n-k$. Thus, $\alpha_i \ne 0$, $1 \le i \le n$. Therefore, $A = LU$ and $\det(A) \ne 0$.*

Proof. It will be shown that $\alpha_i \ne 0$, $1 \le i \le n$.

If $k = 2$, then $|\alpha_1| = |d_1| > |a_1| \ge 0$. Thus, $\alpha_1 \ne 0$ and $|\gamma_1| < 1$. In this way, since $b_2 \ne 0$, we have that $|\alpha_2| = |d_2 - b_2\gamma_1| \ge |d_2| - |\gamma_1||b_2| > |d_2| - |b_2| \ge |a_2| \ge 0$. Hence, $\alpha_2 \ne 0$ and $|\gamma_2| < 1$. If $k > 2$, then $|\alpha_{k-1}| = |d_{k-1} - b_{k-1}\gamma_{k-2}| \ge |d_{k-1}| - |\gamma_{k-2}||b_{k-1}| \ge |d_{k-1}| - |b_{k-1}| > |a_{k-1}| \ge 0$. Thus, $\alpha_{k-1} \ne 0$ and $|\gamma_{k-1}| < 1$. In this way, since $b_k \ne 0$, we obtain that $|\alpha_k| = |d_k - b_k\gamma_{k-1}| \ge |d_k| - |\gamma_{k-1}||b_k| > |d_k| - |b_k| \ge |a_k| \ge 0$. Hence, $\alpha_k \ne 0$ and $|\gamma_k| < 1$.

In order to prove by induction, suppose that $|\gamma_{k+j}| < 1$, $\alpha_{k+j} \neq 0, \forall j, 0 \leq j \leq m$. Thus, for $M = k+m+1$ we have that $b_M \neq 0$, $\gamma_{M-1} < 1$ and $|\alpha_M| = |d_M - b_M\gamma_{M-1}| \geq |d_M| - |\gamma_{M-1}||b_M| > |d_M| - |b_M| \geq |a_M| \geq 0$. Hence, $\alpha_M \neq 0$ and $|\gamma_M| < 1$.

Therefore, by mathematical induction, it is possible to conclude that $\alpha_i \neq 0$, $1 \leq i \leq n$. $\square$

3 SYMMETRIC TRIDIAGONAL TOEPLITZ MATRICES - MAIN THEOREM

In this section, we will study a particular set of symmetric tridiagonal matrices. It will be shown that the matrices belonging to this set are invertible and have LU decomposition. To show a matrix has an LU decomposition, we will prove that its principal minors are not null. This technique is different from one used in the previous section, that was based on parameters from Crout's decomposition given by Equation (2.2).

The matrix A that we will study in this section is a symmetric tridiagonal matrix of order n, whose elements belonging to the main diagonal are equal to d ($A_{ii} = d \neq 0$), and all elements belonging to the lower and upper diagonals are equal to $a \neq 0$. We want to prove that, under certain conditions, the principal minors of A are not null, regardless of the matrix order.

The notation M_k indicates the value of the principal minor of the matrix A mentioned before. Note that $M_1 = d$ and $M_2 = d^2 - a^2$. We will show that $M_k = M_k(d)$ is a polynomial of degree k. Furthermore, if $0 < |d| < 2|a|$ and $\frac{|d|}{|a|} \in \mathbb{Q}$, with $|d| \neq |a|$, then $M_k(d) \neq 0, \forall k$, $1 \leq k \leq n$.

Using Laplace Expansion it is easy to show that

$$M_k = d\,M_{k-1} - a^2 M_{k-2}, \forall\, k > 2. \tag{3.1}$$

Next, we show that $M_k = M_k(d)$ is a monic polynomial of degree k, $M_{2k-1}(d)$ is an odd function, and $M_{2k}(d)$ is an even function.

Proposition 3.1. *For all $k \in \mathbb{N}$, (i) $M_k = M_k(d)$ is a monic polynomial of degree k, i.e., the coefficient of d^k is equal to 1; (ii) $M_{2k-1}(-d) = -M_{2k-1}(d)$ and $M_{2k}(-d) = M_{2k}(d)$.*

Proof. The demonstration is based on mathematical induction. Firstly, note that $M_1(d) = d$ and $M_2(d) = d^2 - a^2$ are monic polynomial of degrees 1 and 2, respectively. Additionally, $M_1(-d) = -d = -M_1(d)$ and $M_2(-d) = (-d)^2 - a^2 = M_2(d)$. Therefore, $M_1(d)$ is an odd polynomial function and $M_2(d)$ is an even polynomial function. Suppose that $M_m(d)$ is a monic polynomial of order m, for every m, with $2 < m < k$. In this way, considering $m = k$ and Equation (3.1), we have that $M_k(d) = d\,M_{k-1}(d) - a^2 M_{k-2}(d)$. Hence, $M_k(d)$ is a monic polynomial of degree k. Now, suppose that M_{2m-1} is an odd polynomial function and M_{2m} is an even polynomial function, for every m, $1 \leq m < k$. If $m = k$, then $M_{2k-1}(-d) = -d\,M_{2k-2}(-d) - a^2 M_{2k-3}(-d) = -(d\,M_{2k-2}(d) - a^2 M_{2k-3}(d)) = -M_{2k-1}(d)$. Hence,

$$M_{2k}(-d) = -d\,M_{2k-1}(-d) - a^2 M_{2k-2}(-d) =$$
$$d\,M_{2k-1}(d) - a^2 M_{2(k-1)}(d) = M_{2k}(d).$$

$\square$

The first property of the polynomial $M_k(d)$ follows easily from Proposition 3.1, as we see in the next corollary.

Corollary 3.1. *If the polynomial $M_k(d)$ has a real root r, then $-r$ is also a real root of this polynomial. Additionally, if k is an odd number, then $M_k(0) = 0$.*

Proof. Based on Proposition 3.1, if $k = 2m$, then $M_k(-r) = M_{2m}(-r) = M_{2m}(r) = M_k(r) = 0$. If $k = 2m-1$, then $M_k(-r) = M_{2m-1}(-r) = -M_{2m-1}(r) = -M_k(r) = 0$. Besides, since $M_{2m-1}(d)$ is a continuous odd function, it follows that

$$\lim_{d\to 0} M_{2m-1}(-d) = -\lim_{d\to 0} M_{2m-1}(d) = -M_{2m-1}(0)$$

and

$$\lim_{d\to 0} M_{2m-1}(-d) = M_{2m-1}(-\lim_{d\to 0} d) = M_{2m-1}(0).$$

Therefore, $M_{2m-1}(0) = 0$. $\square$

In the next result we are supposing that the polynomial $M_k(d)$ may have complex roots, $z = c + bi$ and $\bar{z} = c - bi$. In this case, $Q(d) = (d - z)(d - \bar{z})$ will be a positive quadratic factor of that polynomial.

Based on Proposition 3.1 and Corollary 3.1, we can obtain a particular factorization of the monic polynomial $M_k(d)$. This is shown in the next corollary.

Corollary 3.1. *Let $r_1, r_2, \cdots, r_l$ be the positive real roots of the polynomial $M_k(d)$, with the quadratic factors represented by $Q_1(d), Q_2(d), \cdots, Q_p(d)$, and $2(l+p) = k$, if k is an even number, and $2(l+p)+1 = k$, if k is an odd number. Therefore, that polynomial has the following factorization:*

(i) $M_k(d) = (d^2 - r_1^2)\cdots(d^2 - r_l^2)\,Q_1(d)\cdots Q_p(d)$, if k is an even number;

(ii) $M_k(d) = d\,(d^2 - r_1^2)\cdots(d^2 - r_l^2)\,Q_1(d)\cdots Q_p(d)$, if k is an odd number.

Proof. According to the Fundamental Theorem of Algebra, $M_k(d) = (d-u_1)(d-u_2)\cdots(d-u_k)$, where u_i, $1 \le i \le k$, are the roots of the polynomial $M_k(d)$. By Corollary 3.1, if there is $I, 1 \le I \le k$, such that $u_I > 0$, then u_I and $-u_I$ are roots of $M_k(d)$. In this way, the product $(d-u_I)(d+u_I) = (d^2 - u_I^2)$ appears in the factorization of $M_k(d)$ into linear factors. Moreover, if there is $J, 1 \le J \le k$, such that u_J is a complex root, then the positive quadratic factor $Q_J = (d-u_J)(d-\overline{u_J})$ appears in the factorization of $M_k(d)$. Finally, if k is an odd number, then, by Corollary 3.1, 0 is a root of $M_k(d)$ and, therefore, d is one of the factors of this polynomial. $\square$

The Proposition 3.2 and Proposition 3.3 are going to be useful to prove important properties on the modified Chebyshev polynomial (see the Remark after the Propositon 3.3). These properties are presented in Proposition 3.4, Proposition 3.5, and Corollary 3.1.

Proposition 3.2. *For every $k \in \mathbb{N}$, we have that:*

(i) $M_{2k}(0) = (-1)^k\,|a|^{2k}$;

(ii) $\left[\dfrac{1}{d}\,M_{2k+1}(d)\right]_{|d=0} = (-1)^k\,|a|^{2k}(k+1)$.

Proof. (i) Note that $M_2(d) = d^2 - a^2$. Hence, $M_2(0) = -a^2 = (-1)^1\,|a|^2$. Using mathematical induction, suppose that $M_{2m}(0) = (-1)^m\,|a|^{2m}, \forall m, 1 \leq m < k$. According to Equation (3.1), if $m = k$, we have that

$$M_{2k}(d) = d\,M_{2k-1}(d) - a^2\,M_{2k-2}(d).$$

In this way, $M_{2k}(0) = -|a|^2\,(-1)^{k-1}\,|a|^{2k-2} = (-1)^k\,|a|^{2k}$.

(ii) Note that $M_1(d) = d$ and $M_3(d) = d\,M_2(d) - a^2 M_1(d) = d\,(M_2(d) - a^2)$. Therefore,

$$\frac{1}{d}M_3(d) = M_2(d) - a^2. \text{ Thus, } \left[\frac{1}{d}M_3(d)\right]_{|d=0} = (-1)^1|a|^2 - |a|^2 = (-1)^1\,2\,|a|^2.$$

By induction, suppose that

$$\left[\frac{1}{d}\,M_{2m+1}(d)\right]_{|d=0} = (-1)^m\,|a|^{2m}(m+1), \forall m, 1 \leq m < k.$$

In this way, if $m = k$, we have that $M_{2k+1}(d) = d\,M_{2k}(d) - a^2\,M_{2k-1}(d)$. Hence,

$$\left[\frac{1}{d}\,M_{2k+1}(d)\right]_{|d=0} = M_{2k}(0) - |a|^2\,\left[\frac{1}{d}\,M_{2k-1}(d)\right]_{|d=0}.$$

Thus,

$$\left[\frac{1}{d}M_{2k+1}(d)\right]_{|d=0} = (-1)^k|a|^{2k} - |a|^2(-1)^{k-1}|a|^{2(k-1)}(k-1+1) = (-1)^k|a|^{2k}\,(k+1).$$

$\square$

Proposition 3.3. *Every root of the polynomial $M_k(d)$ can be represented as $|a|x$, for some $x \in \mathbb{C}$.*

Proof. For $k = 1$, note that $M_1(d) = 0 \iff d = 0$. Thus, the root of this polynomial is $u_1 = 0 = |a|.0$. For $k = 2$, note that $M_2(d) = 0 \iff d^2 - |a|^2 = 0$. Hence, the roots of this polynomial are $u_1 = |a|.(1)$ and $u_2 = |a|.(-1)$. Suppose that M_{k-1} has roots given by $|a|u_i$, $1 \leq i \leq k-1$ and that M_{k-2} has roots given by $|a|v_j$, $1 \leq j \leq k-2$. Thus, $M_k(d) = 0 \iff d\,M_{k-1}(d) - a^2\,M_{k-2}(d) = 0$. Therefore, $M_k(|a|x) = 0$ if, and only if, x is root of the following polynomial of degree k: $p_k(x) = x\,(x-u_1)\,(x-u_2)\cdots(x-u_{k-1}) - (x-v_1)\,(x-v_2)\cdots(x-v_{k-2})$. $\square$

Remark: The last polynomial can be defined as $p_k(x) = \frac{1}{|a|^k}\,M_k(|a|x)$. If we consider $p_0(x) = 1$ and $p_1(x) = x$, then $p_k(x)$ is the modified Chebyshev polynomial that was used in the Bank's paper [2]. Next, we are going to prove some of its properties.

Proposition 3.4. *$p_k(x)$ is a monic polynomial and has integer coefficients. Additionally, $p_k(x) = x\,p_{k-1}(x) - p_{k-2}(x), \forall k, k > 2$.*

Proof. Note that $p_1(x) = \dfrac{1}{|a|} M_1(|a|x) = x$ and $p_2(x) = \dfrac{1}{|a|^2} M_2(|a|x) = x^2 - 1$. Thus, $p_k(x)$ is a monic polynomial of degree k, $k \in \{1, 2\}$, with integer coefficients. Consider Equation (3.1) and Proposition 3.1, and suppose that $p_m(x)$ is a monic polynomial of order m with integer coefficients, for every m, $1 \le m < k$, where $k > 2$. If $m = k$, then $p_k(x) = x\,p_{k-1}(x) - p_{k-2}(x)$, because

$$M_k(|a|x) = |a|x M_{k-1}(|a|x) - a^2 M_{k-2}(|a|x), \forall k > 2.$$

Therefore, $p_k(x)$ is also a monic polynomial with integer coefficients. □

Proposition 3.5. *For every $k \in \mathbb{N}$, we have that:*

(i) $p_{2k}(0) = (-1)^k$*;*

(ii) $\left[\dfrac{1}{x} p_{2k+1}(x)\right]_{|x=0} = (-1)^k (k+1)$.

Proof. We are going to use Proposition 3.2 and the definition of the polynomial $p_k(x)$. Note that:

(i) $\quad p_{2k}(x) = \dfrac{1}{|a|^{2k}} M_{2k}(|a|x) \to p_{2k}(0) = \dfrac{1}{|a|^{2k}} M_{2k}(0);$

(ii) $\quad \dfrac{1}{x} p_{2k+1}(x) = \dfrac{1}{x} \dfrac{1}{|a|^{2k+1}} M_{2k+1}(|a|x) = \dfrac{1}{|a|^{2k}} \left[\dfrac{1}{|a|x} M_{2k+1}(|a|x)\right] \to$

$$\left[\frac{1}{x} p_{2k+1}(x)\right]_{|x=0} = \frac{1}{|a|^{2k}} \left[\frac{1}{d} M_{2k+1}(d)\right]_{|d=0}, \text{ where } d = |a|x.$$ □

The next result is a corollary of Proposition 3.5.

Corollary 3.1. *The positive rational roots of the polynomials p_{2k}, if they exist, are equal to* 1 *and the positive rational roots of the polynomials p_{2k+1}, if they exist, must be divisors of $k+1$.*

Proof. If m/q is a rational root of a monic polynomial of degree n, $P_n(x) = \theta_n x^n + \theta_{n-1} x^{n-1} + \cdots + \theta_1 x + \theta_0$, where $\theta_i \in \mathbb{Z}, 0 \le i \le n$ and $mdc(m,q) = 1$, then $m|\theta_0$ and $q|\theta_n$. Since $\theta_n = 1$, because the polynomial is monic, it follows that $|q| = 1$. According to Proposition 3.5, the coefficient θ_0 of a polynomial $p_{2k}(x)$ is equal to $(-1)^k$ and the coefficient θ_0 of a polynomial $x^{-1} p_{2k+1}(x)$ is equal to $(-1)^k (k+1)$ (observe that $p_{2k+1}(0) = 0$, by Corollary 3.1). Therefore, the only possible positive rational root of p_{2k} is 1 and the only possible positive rational roots of p_{2k+1} must be divisors of $k+1$. □

Next, we will show that the roots of the polynomial $p_k(x) = |a|^{-k} M_k(|a|x)$ are real numbers. Additionally, the positive roots belong to the interval $(0, 2)$. Before the presentation of the result, the following two propositions will be demonstrated.

The first propositon below is going to guarantee that $p_k(2) > 0, \forall k \in \mathbb{N}, k > 1$. This property will be employed in Theorem 3.2, where it will be shown that the polynomial p_k has only real roots.

Proposition 3.6. *For every $k \in \mathbb{N}, k > 1$, $M_k(2|a|) = (2k-1)|a|^k$.*

Proof. Since $M_2(d) = d^2 - a^2$, it follows that $M_2(2|a|) = 4|a|^2 - |a|^2 = 3|a|^2$. Now, suppose that $M_m(2|a|) = (2m-1)|a|^m, \forall m, 2 \leq m < k$. According to Equation (3.1), if $m = k$, we have that $M_k(d) = d\,M_{k-1}(d) - a^2 M_{k-2}(d)$. Thus, by induction hypothesis,

$$M_k(2|a|) = 2|a|\,[(2k-3)|a|^{k-1}] - |a|^2\,[(2k-5)|a|^{k-2}] = (2k-1)|a|^k.$$

$\square$

Remark: According to Proposition 3.6, $p_k(2) = |a|^{-k} M_k(2|a|) = (2k-1) > 0$, if $k \in \mathbb{N}$, and $k > 1$.

The next proposition presents a particular factorization of the polynomial $p_k(x)$. This factorization is based on Corollary 3.1 and Proposition 3.3.

Proposition 3.7. *Suppose that $x_1, x_2, \cdots, x_l$ are positive roots of the polynomial $p_k(x)$ and $2l = k$, if k is an even number, and $2l+1 = k$, if k is an odd number. Therefore, this polynomial has the following factorization*

$$(i)\ p_k(x) = (x^2 - x_1^2)\cdots(x^2 - x_l^2), \text{ if } k \text{ is an even number;}$$

$$(ii)\ p_k(x) = x\,(x^2 - x_1^2)\cdots(x^2 - x_l^2), \text{ if } k \text{ is an odd number.}$$

Proof. We just use the definition of the polynomial, $p_k(x) = |a|^{-k} M_k(|a|x)$, and the results were presented in Corollary 3.1 and Proposition 3.3. $\square$

The notation for positive roots of the polynomial $p_k(x)$ is presented in the next definition, and it will be employed in Theorem 3.2.

Definition 3.1. *If the polynomial $p_{2k}(x)$ (or $p_{2k+1}(x)$) has k positive roots in ascending order belonging to the interval $(0,2)$, then they are denoted by $x_{2k}^{(j)}$, $1 \leq j \leq k$ (or $x_{2k+1}^{(i)}$, $1 \leq i \leq k$). We consider by convention $x_1^{(0)} = 0$.*

The next theorem states that all of the roots of the polynomial $p_k(x)$ are real numbers.

Theorem 3.2. *The polynomials $p_{2m}(x)$ and $p_{2m+1}(x)$ have exactly m positive roots belonging to the interval $(0,2)$, for every $m \in \mathbb{N}$, $m \geq 1$. Furthermore,*

$$x_{2m}^{(1)} \in (0,\, x_{2m-1}^{(1)}),\ x_{2m}^{(i)} \in (x_{2m-1}^{(i-1)}, x_{2m-1}^{(i)}),\ 2 \leq i \leq m-1,$$
$$x_{2m}^{(m)} \in (x_{2m-1}^{(m-1)},\, 2),\ \forall m, m \geq 2;$$
$$x_{2m+1}^{(j)} \in (x_{2m}^{(j)},\, x_{2m}^{(j+1)}),\ 1 \leq j \leq m-1,\ x_{2m+1}^{(m)} \in (x_{2m}^{(m)},\, 2),\ \forall m, m \geq 1.$$

Proof. Note that $p_2(x) = x^2 - 1$, $p_3(x) = x(x^2 - 2)$, $p_4(x) = x^4 - 3x^2 + 1$ and $p_5(x) = x(x^4 - 4x^2 + 3)$. In this way,

$$x_2^{(1)} = 1; \quad x_3^{(1)} = \sqrt{2}; \quad x_4^{(1)} = \sqrt{\frac{3-\sqrt{5}}{2}} \text{ and } x_4^{(2)} = \sqrt{\frac{3+\sqrt{5}}{2}};$$

$$x_5^{(1)} = 1 \text{ and } x_5^{(2)} = \sqrt{3}.$$

Therefore, all of the positive roots of these polynomials belong to the interval $(0, 2)$. Additionally,

$$x_3^{(1)} \in (x_2^{(1)},\, 2); \quad x_4^{(1)} \in (0,\, x_3^{(1)}) \text{ and } x_4^{(2)} \in (x_3^{(1)},\, 2);$$

$$x_5^{(1)} \in (x_4^{(1)},\, x_4^{(2)}) \text{ and } x_5^{(2)} \in (x_4^{(2)},\, 2).$$

Suppose that the theorem is valid for every $m \in \mathbb{N}$, $2 \le m < k$. We will show that the theorem is still valid for $m = k$. We are going to use the Proposition 3.4 referring to the equality $p_k(x) = x\,p_{k-1}(x) - p_{k-2}(x)$, and the Intermediate Value Theorem (IVT) to prove the results. Firstly, it will be proved that:

$$\text{(I) } x_{2k}^{(1)} \in (0,\, x_{2k-1}^{(1)}), \quad \text{(II) } x_{2k}^{(i)} \in (x_{2k-1}^{(i-1)},\, x_{2k-1}^{(i)}),\ 2 \le i \le k-1,$$

(III) $x_{2k}^{(k)} \in (x_{2k-1}^{(k-1)},\, 2)$. Then, it will be proved that

$$\text{(IV) } x_{2k+1}^{(j)} \in (x_{2k}^{(j)}, x_{2k}^{(j+1)}),\ 1 \le j \le k-1, \text{ and (V) } x_{2k+1}^{(k)} \in (x_{2k}^{(k)},\, 2).$$

Proof of the item I. Considering Proposition 3.5 (item *i*), we have that $p_{2k}(0) = (-1)^k$. Moreover,

$$p_{2k}(x_{2k-1}^{(1)}) = -\,p_{2k-2}(x_{2k-1}^{(1)}),$$

because $x_{2k-1}^{(1)}$ is a root of $p_{2k-1}(x)$. Note that, by the hypothesis of induction, $x_{2k-2}^{(1)} < x_{2k-1}^{(1)} < x_{2k-2}^{(2)}$. Thus, according to Proposition 3.7 (item *i*), the sign of $p_{2k}(x_{2k-1}^{(1)})$ is determined by

$$sgn(p_{2k}(x_{2k-1}^{(1)})) = -\,(-1)^{k-2} = (-1)^{k-1}.$$

In this way, $sgn(p_{2k}(0)\,p_{2k}(x_{2k-1}^{(1)})) = (-1)^k\,(-1)^{k-1} = -1$. Therefore,

$$p_{2k}(0)\,p_{2k}(x_{2k-1}^{(1)}) < 0.$$

According to IVT, there is a root of the polynomial $p_{2k}(x)$ which belongs to the interval $(0,\, x_{2k-1}^{(1)})$, and it will be denoted by $x_{2k}^{(1)}$.

Proof of the item II. Note that

$$p_{2k}(x_{2k-1}^{(i-1)}) = -\,p_{2k-2}(x_{2k-1}^{(i-1)}), \quad p_{2k}(x_{2k-1}^{(i)}) = -\,p_{2k-2}(x_{2k-1}^{(i)}).$$

Using the hypothesis of induction,

$$x_{2k-1}^{(i-1)} \in (x_{2k-2}^{(i-1)}, x_{2k-2}^{(i)}),\ 2 \leq i \leq k-1, \quad x_{2k-1}^{(i)} \in (x_{2k-2}^{(i)}, x_{2k-2}^{(i+1)}),\ 1 \leq i \leq k-2.$$

Thus, according to Proposition 3.7 (item *i*),

$$sgn(p_{2k}(x_{2k-1}^{(i-1)})) = -\,(-1)^{k-i} = (-1)^{k-i+1},$$
$$sgn(p_{2k}(x_{2k-1}^{(i)})) = -\,(-1)^{k-i-1} = (-1)^{k-i}.$$

In this way,

$$sgn(p_{2k}(x_{2k-1}^{(i-1)})\,p_{2k}(x_{2k-1}^{(i)})) = (-1)^{k-i+1}\,(-1)^{k-i} = (-1)^{2(k-i)+1} = -1.$$

Thus, $p_{2k}(x_{2k-1}^{(i-1)})\,p_{2k}(x_{2k-1}^{(i)}) < 0$. Therefore, according to the IVT, there is a root of polynomial $p_{2k}(x)$ in each interval $(x_{2k-1}^{(i-1)}, x_{2k-1}^{(i)})$. These roots will be denoted by $x_{2k}^{(i)}$, $2 \leq i \leq k-1$.

Proof of the item III. Note that

$$p_{2k}(x_{2k-1}^{(k-1)}) = -\,p_{2k-2}(x_{2k-1}^{(k-1)}),$$

and $p_{2k}(2) = 4k-1 > 0$, according to the remark after Proposition 3.6. Using the hypothesis of induction, we have that $x_{2k-1}^{(k-1)} > x_{2k-2}^{(k-1)}$. Hence, by Proposition 3.7 (item *i*), $sgn(p_{2k}(x_{2k-1}^{(k-1)})) = -1 < 0$. Therefore, according to the IVT, there is a root of polynomial $p_{2k}(x)$, belonging to interval $(x_{2k-1}^{(k-1)}, 2)$. This root will be denoted by $x_{2k}^{(k)}$.

Proof of the item IV. Note that

$$p_{2k+1}(x_{2k}^{(j)}) = -\,p_{2k-1}(x_{2k}^{(j)}) \text{ and } p_{2k+1}(x_{2k}^{(j+1)}) = -\,p_{2k-1}(x_{2k}^{(j+1)}).$$

According to item II,

$$x_{2k}^{(j)} \in (x_{2k-1}^{(j-1)}, x_{2k-1}^{(j)}),\ 2 \leq j \leq k-1,$$
$$x_{2k}^{(j+1)} \in (x_{2k-1}^{(j)}, x_{2k-1}^{(j+1)}),\ 1 \leq j \leq k-2.$$

Thus, using Proposition 3.7 (item *ii*),

$$sgn(p_{2k+1}(x_{2k}^{(j)})) = -\,(-1)^{k-j} = (-1)^{k-j+1},$$
$$sgn(p_{2k+1}(x_{2k}^{(j+1)})) = -\,(-1)^{k-j-1} = (-1)^{k-j}.$$

In this way,

$$sgn(p_{2k+1}(x_{2k}^{(j)})\,p_{2k+1}(x_{2k}^{(j+1)})) = (-1)^{k-j+1}\,(-1)^{k-j} = (-1)^{2(k-j)+1} = -1.$$

Hence, $p_{2k+1}(x_{2k}^{(j)})\, p_{2k+1}(x_{2k}^{(j+1)}) < 0$. Therefore, according to the IVT, there is a root of polynomial $p_{2k+1}(x)$ in each interval $(x_{2k}^{(j)}, x_{2k}^{(j+1)})$. These roots will be denoted by $x_{2k+1}^{(j)}$, $1 \leq j \leq k-1$.

Proof of the item V. Note that

$$p_{2k+1}(x_{2k}^{(k)}) = -\,p_{2k-1}(x_{2k}^{(k)}),$$

and $p_{2k+1}(2) > 0$, according to the remark after Proposition 3.6. Based on item III, $x_{2k}^{(k)} > x_{2k-1}^{(k-1)}$. Thus, according to Proposition 3.7 (item *ii*),

$$sgn(p_{2k+1}(x_{2k}^{(k)})) = -1 < 0.$$

Therefore, according to the IVT, there is a root of polynomial $p_{2k+1}(x)$ which belongs to the interval $(x_{2k}^{(k)}, 2)$. This root will be denoted by $x_{2k+1}^{(k)}$. □

Now we are ready to present our main theorem. Theorem 3.3 yields a simple criterion to identify when a Toeplitz symmetric tridiagonal matrix A is non-singular and has an LU decomposition.

Theorem 3.3. *Let A be a tridiagonal matrix as shown in Equation (2.1). Suppose that A is a Toeplitz symmetric tridiagonal matrix with $d_i = d \neq 0$, $a_i = a \neq 0$, for all i, $1 \leq i \leq n$, and $|d| \neq |a|$. In this way, if $\frac{|d|}{|a|} \in (0,2)$ and $\frac{|d|}{|a|}$ is a rational number, or if $|d| \geq 2|a|$, then A is a non-singular matrix and has an LU decomposition.*

Proof. We are going to prove the first case, where $\frac{|d|}{|a|} \in (0,2)$ and $\frac{|d|}{|a|}$ is a rational number. Thus, taking into consideration Corollary 3.1, Proposition 3.3, and Corollary 3.1, we know that any root of the polynomial $M_k(d)$ can be expressed as $|a|x$, where x is the root of the polynomial $p_k(x) = |a|^{-k} M_k(|a|x)$. Furthermore, the only possible positive rational roots of $p_k(x)$ are positive integers $x \geq 1$. Hence, if $x = \frac{|d|}{|a|} \in (0,2)$ is a rational number different from 1, then $p_k(x) \neq 0$. Hence, $M_k(d) \neq 0, \forall k$, $1 \leq k \leq n$, where $|d| = |a|x$. Therefore, the principal minors of A and $\det(A)$ are not null, regardless of the matrix order.

In the second case, if $|d| > 2|a|$, then A is a strictly diagonally dominant matrix. Hence, A has an LU decomposition and $\det(A) \neq 0$ (see, for example, [4]). Besides, if $|d| = 2|a|$, then, by Theorem 2.1, A is a non-singular matrix and has an LU decomposition. □

4 CONCLUSION

In this work, we have developed new criteria to identify when a Toeplitz symmetric tridiagonal matrix A is non-singular and has an LU decomposition (see Theorem 3.3). These criteria are simple and easy to implement.The main result is the following: "if $0 < |d| < 2|a|$, $|d| \neq |a|$ and $|d|/|a|$ is a rational number, then A has an LU decomposition and $\det(A) \neq 0$", where d is the element that belongs to the main diagonal of A, and a is the element that belongs to both the first diagonal above the main diagonal and the first diagonal below the main diagonal of A. The proof of this result is based on the principle of finite induction and the theory of polynomials. Note that if $|d| > 2|a|$, then A is a strictly diagonally dominant matrix. Hence, by a well-known result, A has an LU decomposition and $\det(A) \neq 0$. Besides, if $|d| = 2|a|$, then, according to Theorem 2.1, A is a non-singular matrix and has an LU decomposition.

We highlight that Toeplitz systems arise in a variety of applications in different fields of mathematics, scientific computing, and engineering (see the Chan and Jin's book [5], from 2007, "An Introduction to Iterative Toeplitz Solvers", the SIAM series on Fundamentals of Algorithms):

- Numerical partial and ordinary differential equations;
- Numerical solution of convolution-type integral equations;
- Statistics—stationary autoregressive time series;
- Signal processing—system identification and recursive filtering;
- Image processing—image restoration;
- Padé approximation—computation of coefficients;
- Control theory—minimal realization and minimal design problems;
- Networks—stochastic automata and neutral networks.

In a future work we are going to present new criteria to identify non-singular tridiagonal and pentadiagonal matrices that admit an LU decomposition. These criteria are simple, easy to implement, and they consider diagonally dominant matrices.

REFERENCES

[1] S.S. Askar & A.A. Karawia. On Solving Pentadiagonal Linear Systems via Transformations. *Mathematical Problems in Engineering*, **2015** (2015). doi:https://doi.org/10.1155/2015/232456.

[2] R.E. Bank & D.J. Rose. Marching algorithms for elliptic boundary value problems. I: The constant coefficient case. *SIAM Journal on Numerical Analysis*, **14**(5) (1977), 792 – 829.

[3] S. Belhaj, F. Hcini, M. Moakher & Y. Zhang. A fast algorithm for solving diagonally dominant symmetric quasi-pentadiagonal Toeplitz linear systems. *Journal of Mathematical Chemistry*, **59** (2021), 757–774. doi:https://doi.org/10.1007/s10910-021-01217-7.

[4] R.L. Burden & J.D. Faires. "Numerical Analysis, 9th ed.". Springer, New York, USA (2014).

[5] R.H.F. Chan & X.Q. Jin. "An Introduction to Iterative Toeplitz Solvers". Fundamentals of Algorithms. SIAM, Philadelphia, USA (2007).

[6] C.M. da Fonseca & J. Petronilho. Explicit inverses of some tridiagonal matrices. *Linear Algebra and its Applications*, **325** (2001), 7 – 21.

[7] M. El-Mikkawy & A. Karawia. Inversion of general tridiagonal matrices. *Applied Mathematics Letters*, **19** (2006), 712 – 720. doi:doi:10.1016/j.aml.2005.11.012.

[8] M.E.A. El-Mikkawy. Notes on linear systems with positive definite tridiagonal coefficient matrices. *Indian Journal of Pure and Applied Mathematics*, **33**(8) (2002), 1285 – 1293.

[9] M.E.A. El-Mikkawy. On the inverse of a general tridiagonal matrix. *Applied Mathematics and Computation*, **150**(3) (2004), 669 – 679. doi:https://doi.org/10.1016/S0096-3003(03)00298-4.

[10] C.F. Fischer & R.A. Usmani. Properties of some Tridiagonal Matrices and their application to Boundary Value Problems. *SIAM Journal on Numerical Analysis*, **6**(1) (1969), 127 – 142. URL `https://www.jstor.org/stable/2156523`.

[11] N.B. Franco. "Cálculo Numérico". Pearson Prentice Hall (2006).

[12] Y. Fu, X. Jiang, Z. Jiang & S. Jhang. Properties of a class of perturbed Toeplitz periodic tridiagonal matrices. *Computational and Applied Mathematics*, **39**(146) (2020). doi:https://doi.org/10.1007/s40314-020-01171-1.

[13] Z. Liu, S. Li, Y. Yin & Y. Zhang. Fast solvers for tridiagonal Toeplitz linear systems. *Computational and Applied Mathematics*, **39**(315) (2020). doi:https://doi.org/10.1007/s40314-020-01369-3.

[14] J.M. McNally. A fast algorithm for solving diagonally dominant symmetric pentadiagonal Toeplitz systems. *Journal of Computational and Applied Mathematics*, **234**(4) (2010), 995 – 1005. doi:https://doi.org/10.1016/j.cam.2009.03.001.

[15] J.M. McNally, L.E. Garey & R.E. Shaw. A split-correct parallel algorithm for solving tridiagonal symmetric Toeplitz systems. *Int. J. Comput. Math.*, **75** (2000), 303–313.

[16] G. Meurant. A review on the inverse of symmetric tridiagonal and block tridiagonal matrices. *SIAM Journal Matrix Anal. Appl.*, **13**(3) (1992), 707 – 728.

[17] P.Z. Revesz. "Introduction to Databases: From Biological to Spatio-Temporal". Springer, New York, USA (2010).

[18] P.Z. Revesz. Cubic spline interpolation by solving a recurrence equation instead of a tridiagonal matrix. In "Proceedings of the First International Conference on Mathematical Methods and Computational Techniques in Science and Engineering". WSEAS Press, Athens, Greece (2014), p. 21–25.

[19] Y. Wei, X. Jiang, Z. Jiang & S. Shon. Determinants and inverses of perturbed periodic tridiagonal Toeplitz matrices. *Advances in Difference Equations*, **2019**(410) (2019). doi:https://doi.org/10.1186/s13662-019-2335-6.

[20] X.L. Zhao & T.Z. Huang. On the inverse of a general pentadiagonal matrix. *Applied Mathematics and Computation*, **202**(2) (2008), 639 – 646. doi:https://doi.org/10.1016/j.amc.2008.03.004.