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A simple organized approach for balancing nuclear equation

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This paper reports on the development of a simple organized approach for balancing a given nuclear equation in which all the reactants and products appear explicitly. A novel general algorithm for finding the balanced form of a nuclear equation has been offered for the said purpose. It has been found that, in addition to having the physically admissible balanced form of a nuclear equation, the present scheme gives birth to a lot of other independent balanced forms of the same nuclear equation, each of which, though theoretically possible, corresponds to unphysical equations. In addition to having novelty, the present scheme is expected to have educational value and it might be of didactic interest to physics students.

Keywords: Chemical reaction; Nuclear reaction; Chemical equation; Nuclear equation; Balanced chemical equation; Balanced nuclear equation.

1. Introduction

Nuclear reactions are of paramount importance from the view point of academic interest and scientific investigations. The approaches used for the understanding of nuclear reactions help a chemist in having deeper insight into chemical reactions. Nuclear reactions also play a vital role in medical diagnosis and treatment.

A nuclear reaction is generally expressed by a nuclear equation, which has got the general form, $\begin{matrix}A_1\\Z_1\end{matrix}$ T + $\begin{matrix}A_2\\Z_2\end{matrix}$ B → $\begin{matrix}A_3\\Z_3\end{matrix}$ R + $\begin{matrix}A_4\\Z_4\end{matrix}$ E, where T is the target nucleus, B the bombarding particle, R the residual product nucleus and E the ejected particle, A_i and Z_i (where i = 1, 2, 3, 4) being the respective mass number and atomic number. Finding correctly balanced equation is essential for the understanding of nuclear reaction. Balanced nuclear equations provide us with excellent clues regarding the energy released in nuclear reactions. Balancing of nuclear equation has to be accomplished by equating the total atomic number as well as the total mass number before and after the reaction by following the laws of conservation of atomic number and mass number respectively in a nuclear reaction. It would be worth mentioning here that the problem of balancing a nuclear equation in which all the reactants and products appear explicitly has never been addressed anywhere in the traditional literature [1-18]. In other words, the traditional literature overlooks such a general problem of balancing a nuclear equation. A simple organized technique to deal with the aforesaid novel problem of balancing a given nuclear equation has been reported in this paper and that in fact resulted from the recognition of [19]. The overall study reveals that, like other mathematical theories [20-22], which are well entertained in Physics, the present scheme gives birth to a lot of independent unphysical balanced forms of a nuclear equation in addition to the one which is only physically admissible in the real world. The proposed scheme is novel and in addition to having educational value, it is mathematically interesting and it will enrich the relevant literature there by enhancing the same as well.

2. Comprehensive review on balancing nuclear equation

In order to demonstrate the fact that the problem of balancing a given nuclear equation in which all the reactants and products appear explicitly (i.e. in which the identity of each of the reactants and products of the given nuclear reaction is known) with all coefficients missing has never been addressed in the traditional literature, a comprehensive review has been made in this section in regard to the various problems addressed in the traditional literature in the area of balancing nuclear equation. In almost all of those problems we are generally provided with a nuclear equation with exactly one coefficient or one of the species (either the reactant or the product) missing. The problem is to identify completely the missing coefficient or the missing species (either the reactant or the product). This very concept of balancing nuclear equation prevails in the traditional literature [1-18] and all problems addressed

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are to be solved making use of the conservation of mass number and charge in the nuclear reaction.

The sample problem considered in [8] is being reproduced below.

Consider the reaction ^{59}Co (p, n). What is the product of the reaction?

$$\begin{matrix}1\\1\end{matrix}\text{H}+\begin{matrix}59\\27\end{matrix}\text{Co}\rightarrow\begin{matrix}1\\0\end{matrix}\text{n}+\begin{matrix}\text{Y}\\\text{X}\end{matrix}\text{Z}$$

On the left side of the equation we have 27 + 1 protons. On the right side we have 0 + X protons where X is the atomic number of the product. Obviously X = 28 (Ni). On the left side, we have 59 + 1 nucleons and on the right side, we must have 1 + Y nucleons where Y = 59. So the product is ^{59}Ni.

The problem considered in Example 29.6 of [9] is quoted below.

"A nuclear reaction of significant note occurred in 1932 when Robert Chadwick in England, bombarded a beryllium target with alpha particles. Analysis of the experiment indicated that the following reaction occurred.

$$\begin{matrix}4\\2\end{matrix}\text{He}+\begin{matrix}9\\4\end{matrix}\text{Be}\rightarrow\begin{matrix}12\\6\end{matrix}\text{C}+\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{X}$$

What is $\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{X}$ in this reaction?"

The above problem has been solved in [9] by balancing mass numbers and atomic numbers on both sides of the said equation to find that $\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{X}=\begin{matrix}1\\0\end{matrix}\text{n}$ (a neutron).

The problem set in Question number 10(i) (Exercise 12) in page number 339 of [10] is of the following type.

Complete the following nuclear reaction:

$$\begin{matrix}35\\17\end{matrix}\text{Cl}+?\rightarrow\begin{matrix}32\\16\end{matrix}\text{S}+\begin{matrix}4\\2\end{matrix}\text{He}$$

The procedure of solution of this type of problem in the traditional literature is to choose first the unknown nucleus/nucleon as $\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{X}$ and then to apply the laws of conservation of electric charge and mass number in the reaction to obtain the result as $\begin{matrix}1\\1\end{matrix}\text{H}$ (a proton).

One of the problems considered in section 15.25 (page number 463) of [11] is as follows.

$$\begin{matrix}6\\3\end{matrix}\text{Li}+\begin{matrix}2\\1\end{matrix}\text{H}\rightarrow\begin{matrix}7\\4\end{matrix}\text{Be}+?$$

This problem has been solved in [11] in the usual way by considering the number of protons and the number of neutrons to be the same on both sides of the equation to obtain the missing figure as $\begin{matrix}1\\0\end{matrix}\text{n}$.

Similar kind of problems as mentioned earlier from [10] and [11] with exactly same procedure of solution have been addressed in [12].

Problem number 3 of Exercises in page 3 of [13] is:

Use the laws of conservation of mass number and charge to determine the identity of X in equations below.

$$\begin{matrix}222\\86\end{matrix}\text{Rn}\rightarrow\begin{matrix}4\\2\end{matrix}\text{He}+\text{X}$$

$$\begin{matrix}14\\6\end{matrix}\text{C}\rightarrow\begin{matrix}0\\-1\end{matrix}\text{e}+\text{X}$$

$$\text{X}\rightarrow\begin{matrix}0\\-1\end{matrix}\text{e}+\begin{matrix}19\\9\end{matrix}\text{F}$$

As before this problem can be solved in the usual way following the traditional technique.

An elaborate video discussion (based exclusively on the laws of conservation of atomic number and mass number in a nuclear reaction equation) for balancing various types of nuclear reaction equations in each of which exactly one of the reactants or products is missing is also available in [14].

Similar type of problem of balancing a nuclear equation in which exactly one of the products or reactants is missing has also been considered in [15] on the basis of the laws of conservation of electric charge and mass number in a nuclear reaction.

Example problem 24.1 found in page 869 of [16] is quoted below.

"Balancing a Nuclear Reaction NASA uses the alpha decay of plutonium-238 ($\begin{matrix}238\\94\end{matrix}\text{Pu}$) as a heat source on spacecraft. Write a balanced equation for this decay."

This problem has been solved in [16] by assuming the reaction equation to be

$$\begin{matrix}238\\94\end{matrix}\text{Pu}\rightarrow\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{X}+\begin{matrix}4\\2\end{matrix}\text{He}$$

and then applying the law of conservation of mass number to find A as 234 along with subsequent application of the law of conservation of charges to get Z = 92, which ultimately identifies the unknown element X as uranium (U) so that the balanced form of the nuclear equation is

$$\begin{matrix}238\\94\end{matrix}\text{Pu}\rightarrow\begin{matrix}234\\92\end{matrix}\text{U}+\begin{matrix}4\\2\end{matrix}\text{He}.$$

The objective Question number 13 in page 1410 of [17] is quoted below.

"In the decay $\begin{matrix}234\\90\end{matrix}\text{Th}\rightarrow\begin{matrix}\text{A}\\\text{Z}\end{matrix}\text{Ra}+\begin{matrix}4\\2\end{matrix}\text{He}$, identify the mass number and the atomic number of the Ra nucleus: (a) A = 230, Z = 92 (b) A = 238, Z = 88 (c) A = 230, Z = 88 (d) A = 234, Z = 88 (e) A = 238, Z = 86."

This problem can also be solved by applying the laws of conservation of mass number and atomic number in the nuclear reaction to find that A = 230, Z= 88, so that (c) is the correct option.

Again Problem number 4 of [18] is being reproduced below.

Identify the missing coefficient in the following nuclear reaction:

$$^{235}{}_{92}U + {}^{1}{}_{0}n \rightarrow \_\_\_\_\_\_ {}^{1}{}_{0}n + {}^{139}{}_{53}I + {}^{95}{}_{39}Y$$

This problem can be solved by making use of the laws of conservation of mass number and charge to identify the missing coefficient as 2.

Thus in the traditional literature, the problem of balancing a nuclear equation is such that it consists of a given nuclear equation in which exactly one coefficient or one of the species (either the reactant or the product) is missing and that missing coefficient or the missing reactant or product could be identified completely and the equation could be balanced by using the laws of conservation of mass number and atomic number in the reaction. But unfortunately the problem of balancing a nuclear equation in which each and every reactant and product appears explicitly with all coefficients missing has never been addressed in the long-running literature [1-18]. A simple organized approach for balancing such a given nuclear equation has been presented in this paper.

3. Underlying principle of the novel method of balancing nuclear equation

The novel general algorithm for balancing a given nuclear equation offered in this paper is based on the following fundamental mathematical fact. If there exist 's' number of different linear simultaneous equations involving (m + n) number of variables $x_1, x_2, ..., x_m, x_{m+1}, ..., x_{m+n}$, then only (m + n - s) number of the variables are independent and hence their values can be assigned arbitrarily to obtain the corresponding values of the remaining 's' number of dependent variables. Thus for example, considering a given nuclear equation with 'm' number of reactants and 'n' number of products, if its balanced form is written by making use of (m + n) coefficients $x_1, x_2, ..., x_m, x_{m+1}, ..., x_{m+n}$, then on equating the total atomic number and total mass number before and after the reaction, it would be possible to have two linear simultaneous equations involving those (m + n) coefficients. Now, if those two linear simultaneous equations are different, then it follows from the above fundamental mathematical fact that, out of the above (m + n) coefficients, only (m + n - 2) number of coefficients will be independent and hence their values can be assigned arbitrarily to obtain corresponding values of the remaining two dependent coefficients for generating the balanced forms of the given nuclear equation.

Let us now make use of the above fundamental mathematical fact to illustrate the procedure of finding balanced form of a given nuclear equation by considering the following examples.

Example 1: Let us consider balancing the following nuclear equation.

$$\begin{matrix}6\\3\end{matrix}\,\mathrm{Li} + \begin{matrix}2\\1\end{matrix}\,\mathrm{H} \rightarrow \begin{matrix}4\\2\end{matrix}\,\mathrm{He}$$

Let us assume that the balanced form of the nuclear equation be

$$(x_1)\begin{matrix}6\\3\end{matrix}\,\mathrm{Li} + (x_2)\begin{matrix}2\\1\end{matrix}\,\mathrm{H} \rightarrow (x_3)\begin{matrix}4\\2\end{matrix}\,\mathrm{He}$$

where $x_i>0$, i ε {1, 2, 3}.

Equating the total atomic number and the total mass number on both sides, we have then,

$$3x_1 + x_2 = 2x_3 \tag{1}$$

$$6x_1 + 2x_2 = 4x_3 \tag{2}$$

Equations (1) and (2) are equivalent to the equation

$$3x_1 + x_2 = 2x_3 \tag{3}$$

In this problem, there are three coefficients (viz. x_1, x_2, x_3) and only one equation involving them. So, here there are in all (3 - 1), i.e. 2 independent coefficients. So, let us now assume that $x_1 = k_1$, and $x_2 = k_2$. Then from equation (3) we get, $x_3 = \frac{3k_1+k_2}{2}$.

Thus we have, $x_1 = k_1$, $x_2 = k_2$, $x_3 = \frac{3k_1+k_2}{2}$.

Now, let $k_1 = 1$, $k_2 = 1$ for which all the x_i s are positive, i ε {1, 2, 3}.

Then in this case we have, $x_1 = 1$, $x_2 = 1$, $x_3 = 2$.

Thus here we have, $x_1 : x_2 : x_3 = 1 : 1 : 2$.

Hence the balanced form of the given nuclear equation can be obtained as

$$\begin{matrix}6\\3\end{matrix}\,\mathrm{Li} + \begin{matrix}2\\1\end{matrix}\,\mathrm{H} \rightarrow 2\begin{matrix}4\\2\end{matrix}\,\mathrm{He}$$

It may be noted that a lot of other balanced forms of the given nuclear equation can be obtained for all other arbitrary choices of the values of k_1 and k_2 for each of which all the x_i s are positive, i ε {1, 2, 3}. But each of such balanced forms of the given nuclear equation must have to be discarded for being unphysical balanced form of the nuclear equation. For example, if $k_1 = \frac{1}{3}$, $k_2 = 1$ so that all the x_i s are positive, i ε {1, 2, 3}, then in this case we have, $x_1 = \frac{1}{3}$, $x_2 = 1$, $x_3 = 1$. It then follows that, $x_1 : x_2 : x_3 = \frac{1}{3} : 1 : 1 = 1 : 3 : 3$.

Thus another balanced form of the given nuclear equation is

$$\begin{matrix}6\\3\end{matrix}\,\mathrm{Li} + 3\begin{matrix}2\\1\end{matrix}\,\mathrm{H} \rightarrow 3\begin{matrix}4\\2\end{matrix}\,\mathrm{He}$$

Although the above balanced form of the given nuclear equation is mathematically possible, it is not at all in compliance with the physics of nuclear reactions and the nuclear reaction mechanism. It is hardly possible for the reactants of the aforesaid balanced nuclear equation to fuse and form a compound nucleus which later fissions under most imaginable condition. Hence such a balanced form of the given nuclear equation must be ruled out for being an unphysical one.

Example 2: Let us proceed for balancing the following nuclear equation.

$$\begin{matrix}27\\13\end{matrix}\,\mathrm{Al} + \begin{matrix}1\\0\end{matrix}\,\mathrm{n} \rightarrow \begin{matrix}26\\13\end{matrix}\,\mathrm{Al} + \begin{matrix}1\\0\end{matrix}\,\mathrm{n}$$

Let the balanced form of the given nuclear equation be

$$(x_1)\ {}^{27}_{13}\mathrm{Al} + (x_2)\ {}^{1}_{0}\mathrm{n} \rightarrow (x_3)\ {}^{26}_{13}\mathrm{Al} + (x_4)\ {}^{1}_{0}\mathrm{n}$$

where $x_i>0$, i ε {1, 2, 3, 4}.

Equating the total atomic number and the total mass number on both sides we get,

$$x_1 = x_3 \tag{4}$$

$$27x_1 + x_2 = 26x_3 + x_4 \tag{5}$$

Here there are in all four coefficients (viz. x_1, x_2, x_3, x_4) and two different equations involving them. Thus in this case there are (4 - 2), i.e. 2 independent coefficients. So, let us now assume that, $x_1 = k_1$, and $x_2 = k_2$. Then from equation (4) we have, $x_3 = k_1$, and from equation (5) we have, $x_4 = k_1 + k_2$.

Thus we have, $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_1$, and $x_4 = k_1 + k_2$.

Now, let $k_1 = 1$, $k_2 = 1$ for which all the x_i s are positive, i ε {1, 2, 3, 4}.

Then we have, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, and $x_4 = 2$

Thus in this case we have, $x_1 : x_2 : x_3 : x_4 = 1 : 1 : 1 : 2$.

Hence the balanced form of the given nuclear equation is given by

$${}^{27}_{13}\mathrm{Al} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{26}_{13}\mathrm{Al} + 2\ {}^{1}_{0}\mathrm{n}$$

Example 3: Let us consider balancing the following nuclear equation.

$${}^{235}_{92}\mathrm{U} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{139}_{56}\mathrm{Ba} + {}^{94}_{36}\mathrm{Kr} + {}^{1}_{0}\mathrm{n}$$

Let the balanced form of the given nuclear equation be

$$(x_1)\ {}^{235}_{92}\mathrm{U} + (x_2)\ {}^{1}_{0}\mathrm{n} \rightarrow (x_3)\ {}^{139}_{56}\mathrm{Ba} + (x_4)\ {}^{94}_{36}\mathrm{Kr} + (x_5)\ {}^{1}_{0}\mathrm{n}$$

where $x_i>0$, i ε {1, 2, 3, 4, 5}.

Equating the total atomic number and the total mass number on both sides, we have,

$$92x_1 = 56x_3 + 36x_4 \tag{6}$$

$$235x_1 + x_2 = 139x_3 + 94x_4 + x_5 \tag{7}$$

In this problem, we have in all five coefficients (viz. x_1, x_2, x_3, x_4, x_5) and two different equations involving them. Thus in this case, there are in all (5 - 2), i.e. 3 independent coefficients. So, let us now assume that $x_1 = k_1$, $x_2 = k_2$, and $x_3 = k_3$

Then from equation (6) we get, $x_4 = \frac{92k_1 - 56k_3}{36}$.

Also from equation (7) we get after a little bit of simplification, $x_5 = \frac{-188k_1 + 36k_2 + 260k_3}{36}$. Thus we have, $x_1 = k_1$, $x_2 = k_2$, $x_3 = k_3$, $x_4 = \frac{92k_1 - 56k_3}{36}$, $x_5 = \frac{-188k_1 + 36k_2 + 260k_3}{36}$. Now, let $k_1 = 1$, $k_2 = 1$, $k_3 = 1$ for which all the x_i s are positive, i ε {1, 2, 3, 4, 5}. Then in this case we have, $x_1 = 1$, $x_2 = 1$, $x_3 = 1$, $x_4 = \frac{92-56}{36} = 1$, $x_5 = \frac{-188+36+260}{36} = 3$. Thus we have, $x_1 : x_2 : x_3 : x_4 : x_5 = 1 : 1 : 1 : 1 : 3$.

Hence the balanced form of the given nuclear equation can be obtained as

$${}^{235}_{92}\mathrm{U} + {}^{1}_{0}\mathrm{n} \rightarrow {}^{139}_{56}\mathrm{Ba} + {}^{94}_{36}\mathrm{Kr} + 3\ {}^{1}_{0}\mathrm{n}$$

Each of the balanced forms found in the above examples corresponds to physically acceptable balanced form, which has been obtained for particular assignment(s) of values to relevant k_i s. For all other choices of the values of those k_i s, other types of balanced forms of the nuclear equation (which are unphysical) will result in each of the above examples.

The procedure employed in the above examples for finding the balanced form of a nuclear equation leads to the following novel general algorithm.

4. Novel general algorithm for balancing a nuclear equation

Step 1: After writing down the balanced nuclear equation using the coefficients $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ where "m" and "n" are respectively the total number of reactants and products of the given nuclear equation, obtain the set of two different simultaneous linear equations involving the coefficients $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_{m+n}$ by making use of the principles of conservation of atomic number and mass number before and after the reaction.

Step 2: Assign the (m + n - 2) independent coefficients $x_1, x_2, \ldots, x_m, \ldots, x_{m+n-2}$, the values $k_1, k_2, \ldots, k_{m+n-2}$ respectively, where $k_i>0$, i = 1, 2, ..., m, ..., m + n - 2, and express the remaining two dependent coefficients x_{m+n-1}, and x_{m+n} in terms of $k_1, k_2, \ldots, k_{m+n-2}$.

Step 3: Assign a particular set of values to $k_1, k_2, \ldots, k_{m+n-2}$, ($k_i>0$), such that all the x_i s involved in the two simultaneous linear equations obtained in Step 1 assume positive values and then obtain the ratio $x_1: x_2 : x_3 : \ldots : x_{m+n}$ in the simplest possible positive integer form, $c_1: c_2: c_3 : \ldots : c_{m+n}$.

Step 4: Use the values $c_1, c_2, c_3, \ldots, c_{m+n}$ (obtained in Step 3) respectively for $x_1, x_2, x_3, \ldots, x_{m+n}$ to write down a balanced form of the nuclear equation under consideration.

5. Discussion

A simple organized technique for finding the balanced form of a given nuclear equation in which the names of all the species (viz. the reactants and the products) are known has been offered in this paper along with a novel general algorithm for the said purpose. Balancing problem of such a nuclear equation in which the identities of all the reactants and products are completely known has never been addressed in the traditional literature [1-18]. To that extent the problem addressed in this paper is being claimed to be novel. Furthermore, as far as the method of balancing of a nuclear equation is concerned, the technique offered in this paper is also being claimed to be novel. It has been found that the novel general algorithm presented gives birth to a lot of balanced forms of a nuclear equation. Out of all such balanced forms, only one will be physically admissible (which may result from more than one type of assignment of values to k_1, k_2, ..., k_{m+n-2}, ($k_i > 0$)) and all other generated balanced forms will correspond to unphysical nuclear reaction equations. This is because the probability that the reactants corresponding to each of those balanced nuclear equations fuse and form a compound nucleus which later fissions is vanishingly small under most imaginable conditions. In other words, which of these situations will arise in nuclear physics, has to be determined by the probability cross section and energy consideration of these reactions. Hence all such balanced forms must have to be discarded for being unphysical balanced equations.

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