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ARTICLES

TWO-DIMENSIONAL REPRESENTATION FOR TWO-STAGE NETWORK DEA MODELS

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ABSTRACT. We propose two-dimensional representations for the efficient frontier for two different Network DEA (NDEA) models. Both representations follow a previous study that developed a generalization of the two-dimensional representation method for the classic DEA using a new form of linearization. The graphical representation of the efficient frontier allows managers and decision-makers unfamiliar with linear programming and DEA to understand the results obtained simply and clearly. In this study, we can obtain graphs that provide information about the efficiencies of each DMU. Furthermore, for the NDEA with an efficiency decomposition approach, it is possible to obtain a graph to evaluate the technical efficiency for each stage, as well as for the overall efficiency. The NDEA model based on efficiency composition only produces one graph. However, this unique representation shows the stage-level and the overall process level information within the same graph.

Keywords: data envelopment analysis, network DEA, two-dimensional representation.

1 INTRODUCTION

Due to its mathematical nature, the results obtained by DEA (Data Envelopment Analysis) and Network DEA (NDEA) models are difficult to interpret for those who are unfamiliar with linear programming. For this reason, graphical representations of the efficient frontier are especially useful to facilitate understanding. In addition, some decision makers may have difficulties with numerical values, and may prefer values and/or verbal judgments (Fasolo and Bana e Costa, 2014) or even graphic presentations.

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When we have only one input and one output, we can plot these data in one graph and obtain the graphical representation of the efficient frontier. However, in the presence of more variables, the graphical representation is not simple. Some studies have proposed methodologies to obtain the graphical representation of the efficient frontier (to view a detailed literature review see Bana e Costa et al. (2016)) but the techniques proposed so far entail some difficulties, as harder visualization with many variables, use of unusual DEA models and a lack of clear depiction of the efficient frontier.

Bana e Costa et al. (2016) proposed a new approach that uses modified values of the virtual input and the virtual output to plot all DMUs. This approach generates the two-dimensional representation of the efficient frontier to standard DEA models which circumvents the disadvantages of previous visualizations and can be used with multiple variables. Applications of this approach can be found in Sow et al. (2016), Reis et al. (2017) and Ferraz et al. (2021).

As for the representation of the efficient frontier in the NDEA, traditionally it only occurs for the internal stages of the DMU, and not for the overall stage. It is performed in the same way as in the classic DEA, only for DMUs with one exogenous input, one intermediate variable, and one exogenous output, but with a rotation of 90 degrees of the axes (Kao & Hwang, 2014). Figure 1 shows the representation of the efficient frontier for the data set in Table 1 in the input-oriented CCR model.

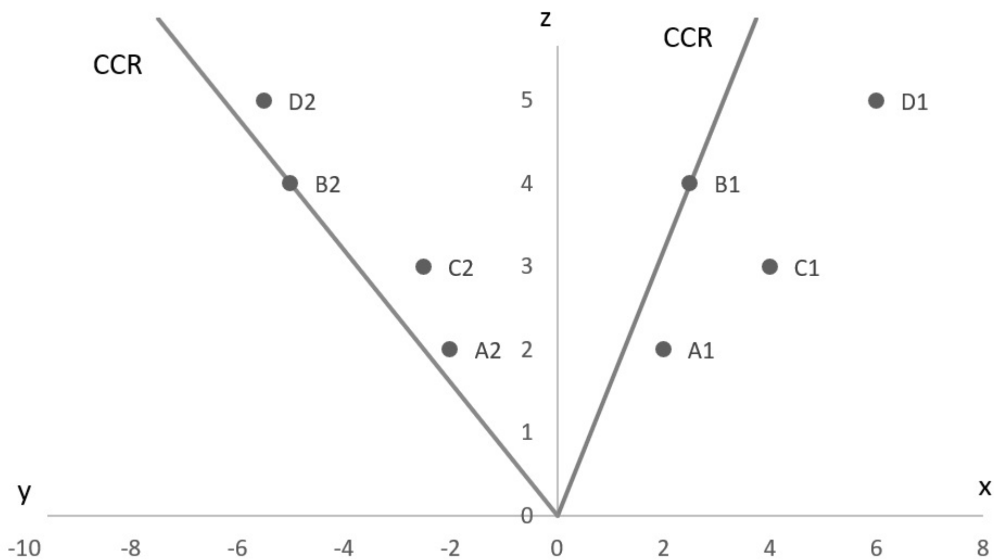

Figure 1 – Graphical representation for the efficient frontier for the NDEA model (Kao & Hwang, 2014).

We can note that the overall efficiency is obtained by multiplying the efficiency of each stage and that DMU B was efficient in both stages. Thus, DMU B is the only efficient one globally and is represented at the efficient frontier in both stages of Figure 1.

Table 1 – Numerical example for the NDEA relational multiplicative model with 4 DMUs.

DMU	*Input*	Int. Var.	*Output*	Eff. Stage 1	Eff. Stage 2	Overall Eff.
A	2	2	2	0,625	0,800	0,500
B	2,5	4	5	1,000	1,000	1,000
C	4	3	2,5	0,469	0,667	0,312
D	6	5	5,5	0,521	0,880	0,458

As we can see, for Network DEA the graphical representation of the efficient frontier is even more restricted or unknown, and there are not many studies on this. Thus, the present study aims to propose a model of graphical representation of the efficient frontier in the presence of multiple variables for the relational multiplicative NDEA model (Kao & Hwang, 2008) and the model developed in Despotis et al. (2016a), generalizing the model proposed in the paper Bana e Costa et al. (2016). The reason why we chose these two NDEA models to develop their two-dimensional graphical representation is that both present a form of linearization similar to the linearization method used in the classic DEA and, therefore, the approach created by Bana e Costa et al. (2016) can be used with some developments. In this study, we will analyze the basic two-stage structure. In this structure, the outputs of the first stage are exclusively the inputs of the second stage as well as the inputs from the second stage are exclusively outputs from the first stage.

The graphical representation of the efficient frontier is extremely valuable as it will show which are the efficient DMUs, which are the inefficient ones, how close to the efficient frontier each DMU is and how much each DMU must improve, reducing your virtual input or increasing the virtual output to become efficient, not only for the overall stage as well as for the internal stages of each DMU. In addition, analyzes can be carried out in relation to each stage and their average efficiencies.

It is important to emphasize that this analysis could be done through the tables obtained with the numerical results of the applied model, however, the two-dimensional representation of the efficiency frontier summarizes this information in a simple and clear way, which can help decision makers, who need to make faster and more accurate decisions not to make mistakes in their actions, because in a world that data has been increasingly abundant, clarity and simplification can be very valuable. In addition, the graphical representation can help DEA students to better understand the learned models.

In the next section, we present a literature review of the model developed in Despotis et al. (2016a) and its proposed two-dimensional graphical representation. This is followed by a review of the relational multiplicative NDEA model, and its two-dimensional graphical representation proposed in Section 3. In Section 4 a numerical example is presented. Finally, Section 5 outlines the conclusions of the paper.

2 THE DESPOTIS ET AL. (2016A) MODEL

In the following subsections, we will present a review of the model presented in Despotis et al. (2016a) and propose a two-dimensional representation of the efficient frontier.

2.1 Review of the Despotis et al. (2016) model

The "composition approach" was proposed in Despotis & Koronakos (2014) to solve the problems of some prior NDEA models point out by themselves. According to the authors, the additive decomposition method (Chen et al., 2009) always favors one of the stages over the other. On the other hand, the efficiency obtained by the relational multiplicative method (Kao & Hwang, 2008) is not unique and requires a post-optimality proceeding to obtain the efficiency values. The "composition approach" received this name because the method first obtains the efficiency value of the intermediate stages, and then aggregates those values to obtain the overall efficiency of the DMU.

Later, Despotis et al. (2016a) developed a model to situate the efficient frontier established by the new approach. For this, the authors defined the efficiency of the first stage as output-oriented as in (1) and the second stage as input-oriented as in (2). The intermediate weights are the same in (1) and in (2).

$$
\begin{aligned}
E_0^1 &= min \frac{\sum_{i=1}^{m} v_i x_{i0}}{\sum_{d=1}^{D} w_d z_{d0}} \\
s.t. &\frac{\sum_{i=1}^{m} v_i x_{ij}}{\sum_{d=1}^{D} w_d z_{dj}} \geq 1, j = 1,2,\ldots,n \\
v_i, w_d &\geq 0; i = 1,2,\ldots,m; d = 1,2,\ldots,D
\end{aligned} \tag{1}
$$

$$
\begin{aligned}
E_0^2 &= max \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{d=1}^{D} w_d z_{d0}} \\
s.t. &\frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{D} w_d z_{dj}} \leq 1, j = 1,2,\ldots,n \\
u_r, w_d &\geq 0; r = 1,2,\ldots,s; d = 1,2,\ldots,D
\end{aligned} \tag{2}
$$

In models (1) and (2) we have n DMUs (j = 1,. . . ,n) and a production process composed of two sub-processes. Also, we have m inputs X_{ij}, i = 1,. . . ,m, s outputs Y_{rj}, r = 1,. . . , s, and D intermediate products Z_{dj}, d = 1,. . . ,D. After some mathematical transformations, Despotis et al. (2016a) proposed to solve models (1) and (2) as a multiobjective linear program (MOLP), as in (3).

$$
\begin{aligned}
E_0^1 &= min \sum_{i=1}^{m} v_i x_{i0} \\
E_0^2 &= max \sum_{r=1}^{s} u_r y_{r0} \\
s.t. &\sum_{d=1}^{D} w_d z_{d0} = 1
\end{aligned}
$$

$$\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1,2,\ldots,n$$

$$\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, j = 1,2,\ldots,n$$

$$u_r, v_i, w_d \geq 0, i = 1,2,\ldots,m, r = 1,2,\ldots,s, d = 1,2,\ldots,D \tag{3}$$

Regarding the model (3), the vector ($E_0^1 \geq 1, E_0^2 \leq 1$) constitutes its ideal point. This point is represented by the highest virtual output and the lowest virtual input obtained from the optimal solution set and it is not generally achievable. Thus, solving (3) as a MOLP means finding several non-dominated feasible solutions. To obtain a unique solution on the Pareto front, Despotis et al. (2016a) employed the Tchebycheff norm. The authors proposed the MinMax model (4), which minimizes the maximum of the deviations of ($\sum_{i=1}^{m} v_i x_{i0}$, $\sum_{r=1}^{s} u_r y_{r0}$) from the ideal point (E_0^1, E_0^2), denotes by δ.

$$
\begin{aligned}
& min \delta \\
& s.t. \ \sum_{i=1}^{m} v_i x_{i0} - \delta \leq E_{jo}^1 \\
& \sum_{r=1}^{s} u_r y_{r0} + \delta \geq E_{jo}^2 \\
& \sum_{d=1}^{D} w_d z_{d0} = 1 \\
& \sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1,2,\ldots,n \\
& \sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, j = 1,2,\ldots,n \\
& u_r, v_i, w_d, \delta \geq 0, i = 1,2,\ldots,m, r = 1,2,\ldots,s, d = 1,2,\ldots,D
\end{aligned} \tag{4}
$$

Figure 2 presents a hypothetical result of applying the model (4). The point F represents the solution of the MinMax model. The solution corresponds to the intersection of the Pareto front (the ABCD line) and a ray from the ideal point. We can see by analyzing Figure 7 that the model presents a unique point on the Pareto front as a solution. This means that the model (4) always provides unique efficiency scores for both intermediate stages (Despotis et al., 2016a), a fact that did not occur in the multiplicative relational model.

After solving the model (4), the efficiency scores for DMU_0 are as in (5) for the first stage and as in (6) for the second stage. We can note that the efficiency of the first stage is equal to the inverse of the exogenous virtual input, and the efficiency of the second stage is equal to the exogenous virtual output.

$$E_0^1 = \frac{\sum_{d=1}^{D} w_d z_{d0}}{\sum_{i=1}^{m} v_i x_{i0}} = \frac{1}{\sum_{i=1}^{m} v_i x_{i0}} \tag{5}$$

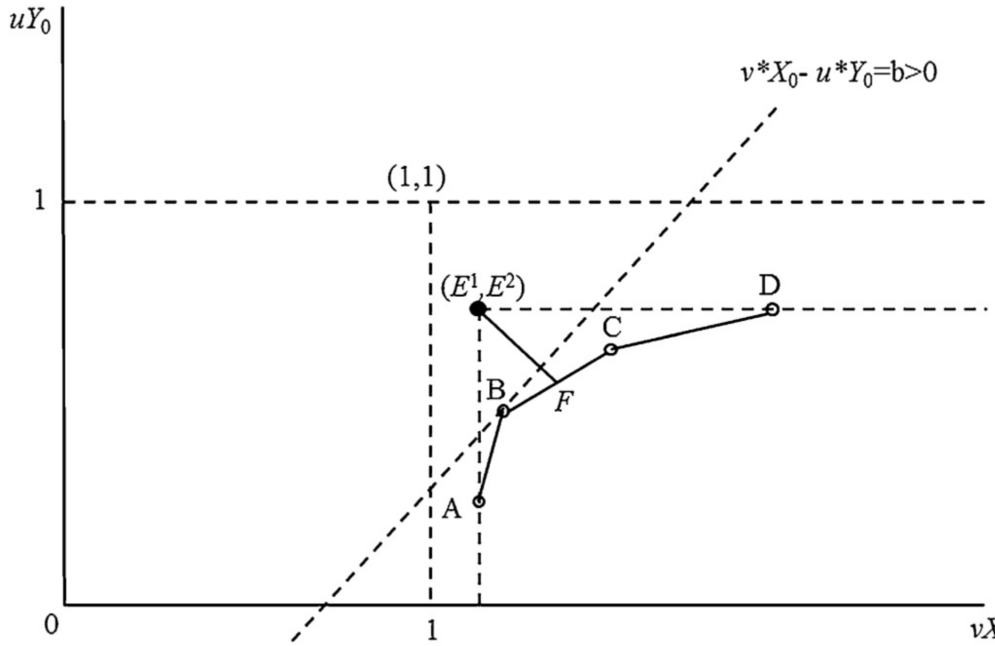

Figure 2 – The Pareto front of (2) and the optimal solution of model (3) Despotis et al. (2016a).

$$E_0^2 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{d=1}^{D} w_d z_{d0}} = \sum_{r=1}^{s} u_r y_{r0} \tag{6}$$

Lastly, the efficiencies of the two individual stages need to be combined to obtain the overall efficiency, which can be obtained by using arithmetic mean as in (7) or by multiplication as in (8).

$$E_0 = \frac{1}{2}\left(E_0^1 + E_0^2\right) \tag{7}$$

$$E_0 = E_0^1 * E_0^2 = \frac{1}{\sum_{i=1}^{m} v_i x_{i0}} * \sum_{r=1}^{s} u_r y_{r0} = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{i=1}^{m} v_i x_{i0}} \tag{8}$$

In the next subsection, we present a model proposal to obtain a two-dimensional graphical representation in the Despotis et al. (2016a) model.

2.2 Two-dimensional representation of the Despotis et al. (2016a) model

The composition approach developed in Despotis et al. (2016a) takes into account that first we need to obtain the efficiency of the internal stages, and then we can aggregate them into the overall efficiency. In this study, we are going to focus on multiplicative aggregation.

Thus, we can obtain the overall efficiency as the division of exogenous virtual output by exogenous virtual input (as seen in 8). The linearization constraint does not interfere with the two-dimensional graph (as seen in 3), and we can represent the overall efficient frontier with the exogenous virtual input as the x-axis and the exogenous virtual output as the y-axis. When these values are equal, the efficiency of the DMU is 1 and we can define the efficient frontier as a 45-degree line, as in Bana e Costa et al. (2016). If the method to obtain the overall efficiency were the arithmetic mean of the internal efficiencies, the overall efficiency would not be obtained by dividing the virtual output by the virtual input, in this way, the distance from the frontier obtained by the graph with the proposed axes, has no relation with the efficiency of the DMU.

Also, as the efficiency of the first stage is equal to the inverse of the exogenous virtual input (as seen in 5), the efficiency of this stage is equal to 1 when the exogenous virtual input is equal to 1. Therefore, we can note that we represent the efficient DMUs in the first stage at the graph where the exogenous virtual input is equal to 1. Analogously, the DMU is only efficient in the second stage if the virtual output equals 1 (as seen in 6). Then, we represent the efficient DMUs in the second stage at the graph where the exogenous virtual output is equal to 1.

In consequence, we should plot DMUs that are efficient in both the first and the second stage at point (1,1) of the graph. These DMUs are also overall efficient and are represented in the two-dimensional frontier, which is a 45-degree line, where the values of the axis are the same.

Another characteristic of this representation is that we can separate DMUs into two groups. The first group is composed of DMUs that are more efficient in the first stage than the second stage, while the second group is the opposite. To determine these groups, we should first find the border where the efficiency of the first stage is equal to the efficiency of the second stage. We can obtain this hyperbola, as in (9).

$$E_0^1 = E_0^2 \rightarrow \frac{1}{\sum_{i=1}^{m} v_i x_{i0}} = \sum_{r=1}^{s} u_r y_{r0} \rightarrow \frac{1}{I} = O \tag{9}$$

It is important to highlight that the hyperbola (9) is not an efficiency frontier, but a dividing line that separates the DMUs according to their efficiency result in each internal stage. It assists in decision making based on the visualization of the graph. By replacing the equal sign in equation (9) with inequalities, we were able to obtain regions where DMUs have a higher efficiency value in an internal stage compared to the other. We represent DMUs that are more efficient in the first than the second stage below the hyperbola, while we represent DMUs that are more efficient in the second stage above the hyperbola.

Both the frontier obtained for the first and second stages and the proposed hyperbola are not related to the aggregation method used to obtain the overall efficiency of a DMU, therefore, they would be valid if the aggregation was done through the arithmetic mean.

We can note that this two-dimensional representation, although unique for the overall process, contains information about the internal stages of the DMUs as well. Thus, we can obtain information from all stages in one two-dimensional representation, differently from the representation proposed for the relational multiplicative model that presents a two-dimensional frontier representation for each stage and another one for the overall process, separately.

We have developed a flowchart, in Figure 3, which presents the necessary steps to obtain a two-dimensional representation of the efficient frontier.

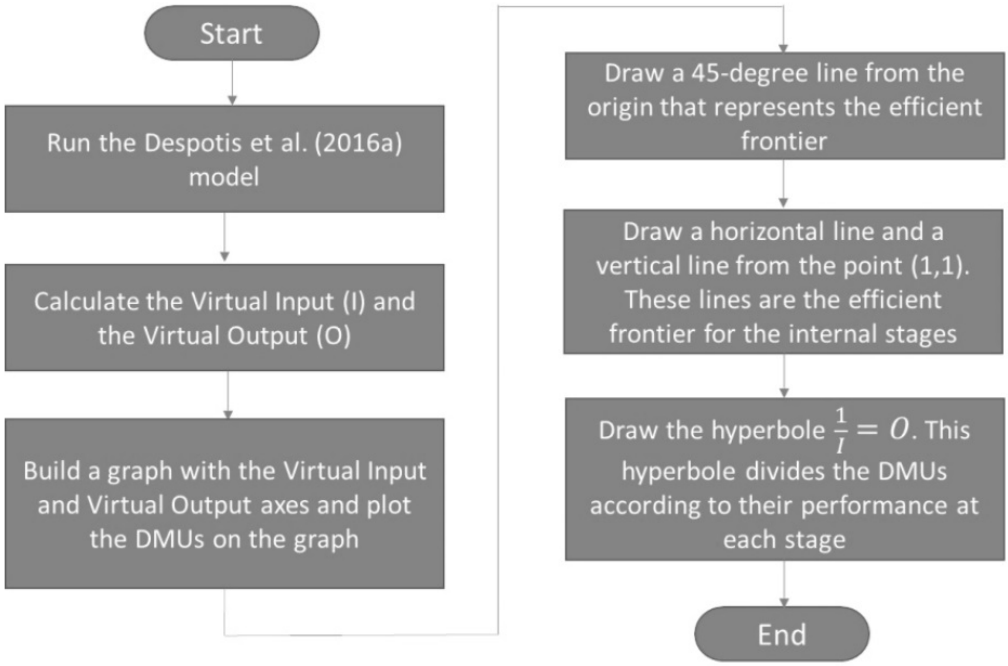

Figure 3 – Flowchart for the development of the efficient frontier for the Despotis et al. (2016a) model.

We have developed a numerical example to demonstrate the construction of the efficient frontier for the overall stage. We have used the data in Table 2 and the two-dimensional representation of the efficient frontier in Figure 4.

Table 2 – Numerical example for the Despotis et al. (2016a) model.

DMUs	Exogenous Input	Intermediate Variable	Exogenous Output	Overall Efficiency	First Stage Efficiency	Second Stage Efficiency	v_1	w_1	u_1
A	2	2	2	0,250	0,500	0,500	1,000	0,500	0,250
B	2	4	8	1,000	1,000	1,000	0,500	0,250	0,125
C	4	4	8	0,500	0,500	1,000	0,500	0,250	0,125
D	6	12	5,5	0,229	1,000	0,229	0,167	0,083	0,042
E	3	4,5	8	0,667	0,750	0,889	0,444	0,222	0,111
F	2,5	4,5	6	0,600	0,900	0,667	0,444	0,222	0,111

We can note that DMU B is the only DMU overall efficient, and it is the only DMU represented at the efficient frontier. Also, we can draw horizontal, or vertical lines from the DMU under analysis to calculate the efficiency of that DMU. This can be analyzed in Figure 4 for DMU A, efficiency can be obtained by dividing $\overline{AA'}/\overline{AA''} = 0.5/2 = 0.250$. In this example, $\overline{AA''}$ corresponds to

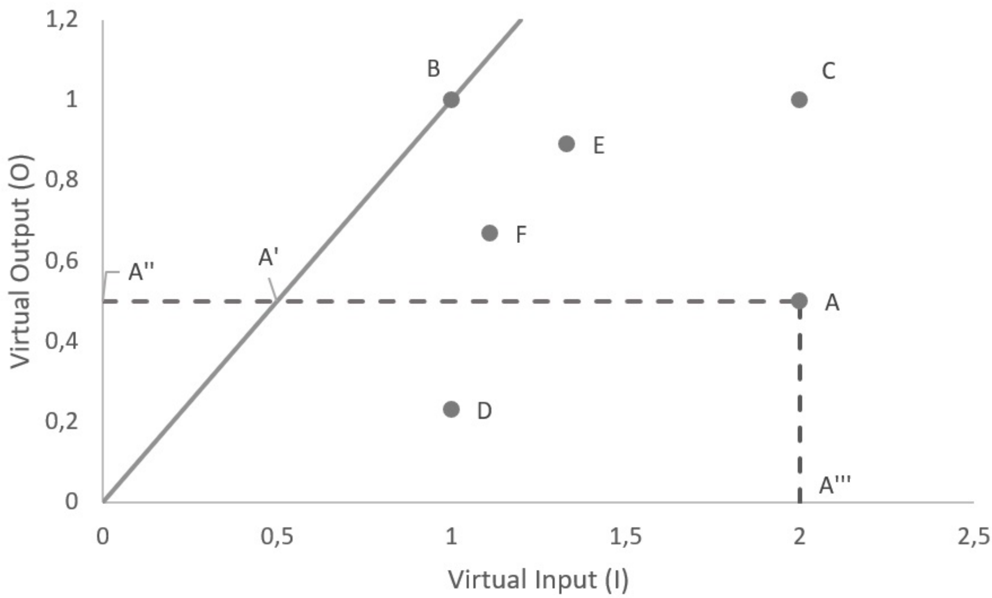

Figure 4 – Two-dimensional representation of the efficient frontier for a numerical example in the Despotis et al. (2016a) model.

the Virtual input as we can see in the graph while $\overline{AA'}$ corresponds to Virtual output, because the distance from the point A to the efficient frontier $\overline{(AA')}$ is the same distance from the point A to the x-axis ($\overline{AA'''}$). This is true because the efficient frontier has an angle of 45 degrees to the efficient frontier. As it uses Virtual Input and Virtual Output as axes of the graph, the logic for calculating efficiency is the same as that used for the model by Bana and Costa et al. (2016).

However, as seen in Figure 5, we can draw efficient frontiers for the internal stages, as well as develop a hyperbola which divides the DMUs according to their performance in each internal stage.

To be efficient globally, the DMU needs to be efficient at all internal stages, so DMU B, which is the only efficient DMU is represented at the intersection of the developed lines. Also, the efficient frontier of the internal stages is represented in dashed lines. For example, DMU B and D are efficient in the first stage and are represented in the vertical dashed line, while DMUs B and C are efficient in the second stage and are represented in the horizontal dashed line.

Also, we have developed another representation, in Figure 6, to show that the closer to the dashed frontier, the more efficient is that DMU at that evaluated stage.

We can obtain the efficiency of DMU F by dividing $\overline{F'F''}/\overline{FF''} = 1/1.111 = 0.900$. This occurs because of the properties of the study model. We saw in equation (5) that the efficiency of the first stage is equal to the inverse of the virtual input, and since the proposed frontier is a vertical line (that is, the distance from the y-axis to the frontier will always be 1), the calculation performed

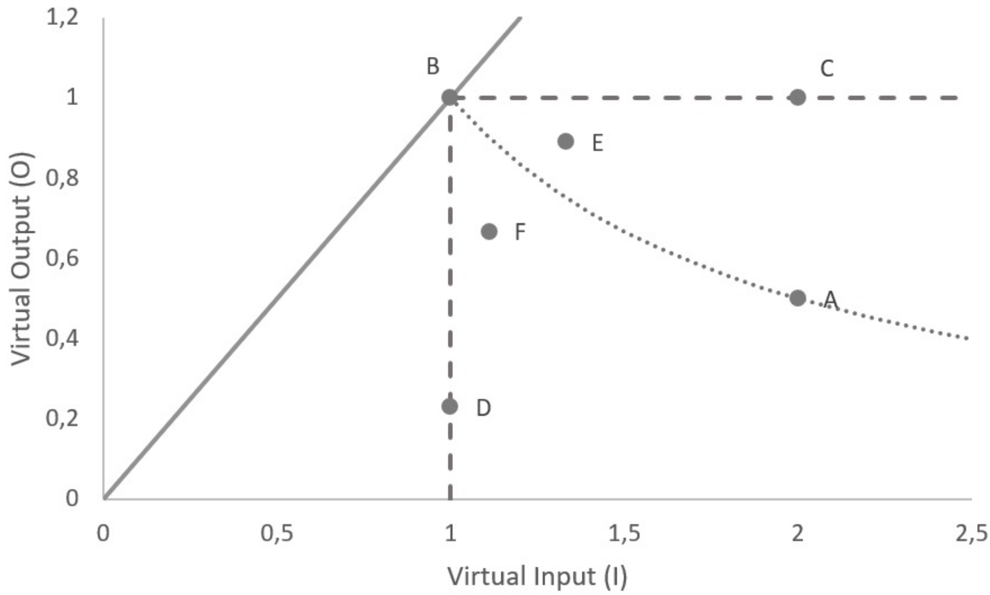

Figure 5 – Two-dimensional representation of the efficient frontier for a numerical example for the Despotis et al. (2016a) model with internal frontiers and the hyperbola.

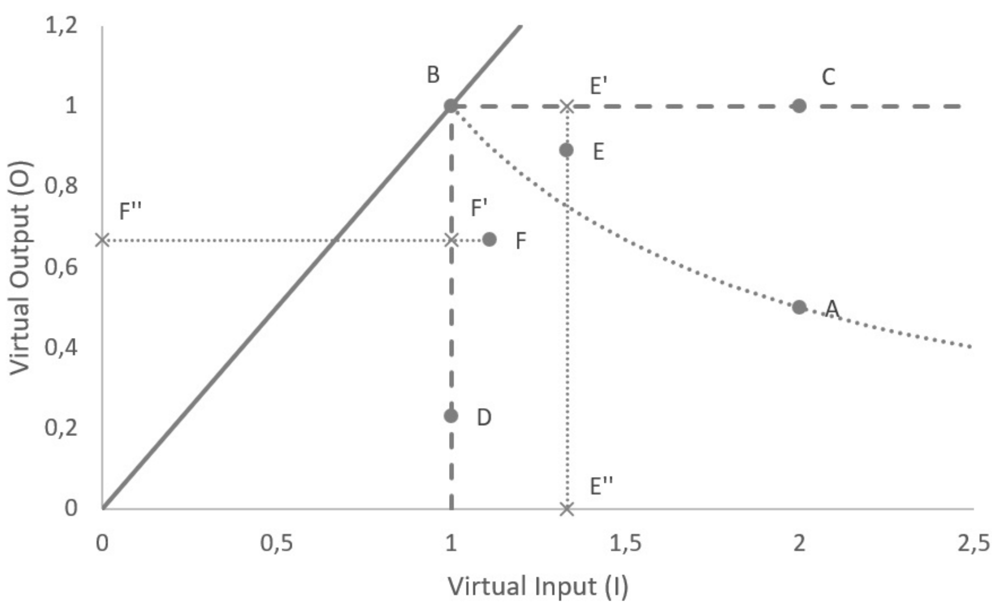

Figure 6 – Two-dimensional representation of the efficient frontier for a numerical example in the Despotis et al. (2016a) model with the efficiency calculation for DMUs E and F.

for DMU F calculates exactly the inverse of the virtual input. This same logic can be used for the second stage, as we can see for DMU E which has an efficiency equal to $\overline{EE''}/\overline{E'E''} = 0.889/1 = 0.889$. In this case, the efficiency is equal to the virtual output (as seen in equation 6), and, since the frontier is a horizontal line that has a distance equal to one for the x-axis, the calculation will always obtain the efficiency for this stage. Therefore, the closer to the frontiers, the more efficient the DMU for that stage.

Finally, the hyperbola drawn by the dotted line, which passes through DMUs A and B, divides DMUs between those that are most efficient in the first stage and those that are most efficient in the second stage, as seen in equation (9). In this case, DMUs D and F are represented below the hyperbola and are more efficient in the first stage than in the second. While DMUs C and E are more efficient in the second stage than in the first. As DMUs A and B are represented in the hyperbola, both have equal efficiency for both internal stages. This representation helps decision-makers, as they can have an overview of the performance of DMUs, for all stages at the same time.

3 RELATIONAL MULTIPLICATIVE NDEA MODEL

In the following subsections, we will present a review of the relational multiplicative model presented in Kao & Hwang (2008) and propose its two-dimensional representation of the efficient frontier.

3.1 Review of Relational Multiplicative NDEA model

The multiplicative relational NDEA model proposed in Kao & Hwang (2008) is based on the CCR model. In (10) we have the linearized model for the CCR input-oriented with two stages in series, without exogenous outputs in the first stage and exogenous inputs in the second stage, where E_o is the overall efficiency of the process, v_i, u_r and w_d are the multipliers of the variables, x, y and, z are the values of the inputs, the outputs and the intermediate variables, respectively.

$$
\begin{gathered}
E_o = max \sum_{r=1}^{s} u_r y_{r0} \\
s.t. \sum_{i=1}^{m} v_i x_{i0} = 1 \\
\sum_{d=1}^{D} w_d z_{dj} - \sum_{i=1}^{m} v_i x_{ij} \leq 0, j = 1,2,\ldots,n \\
\sum_{r=1}^{s} u_r y_{rj} - \sum_{d=1}^{D} w_d z_{dj} \leq 0, j = 1,2,\ldots,n \\
u_r, v_i, w_d \geq 0, i = 1,2,\ldots,m, r = 1,2,\ldots,s, d = 1,2,\ldots,D
\end{gathered}
\tag{10}
$$

From (10) we obtain the overall efficiency and multipliers of each variable. The efficiencies of the internal stages are obtained through equations (11) and (12).

$$E_0^1 = \frac{\sum_{d=1}^{D} w_d z_{d0}}{\sum_{i=1}^{m} v_i x_{i0}} \tag{11}$$

$$E_0^2 = \frac{\sum_{r=1}^{s} u_r y_{r0}}{\sum_{d=1}^{D} w_d z_{d0}} \tag{12}$$

There may be an alternative set of multipliers that optimizes the objective function (10). In this case, the overall efficiency remains the same, however, the efficiencies of the internal stages may change somewhat. So, we should use another objective function that optimizes one of the analyzed stages, according to its importance, as demonstrated in Kao & Hwang (2008).

Finally, we can calculate the overall efficiency by multiplying the efficiency of its internal stages, as in (13). Thus, for a DMU to be efficient, it must be efficient at each stage.

$$E_0 = E_0^1 x E_0^2 \tag{13}$$

In the next subsection, we present a proposal to obtain a two-dimensional graphical representation for the Kao & Hwang (2008) model.

3.2 Two-dimensional representation for the Relational Multiplicative NDEA model

We introduce two-dimensional representation graphs for the relational multiplicative NDEA model proposed in Kao & Hwang (2008) to represent the overall efficiency stage and each internal efficiency stage. We use for this representation the values of the virtual input and the virtual output as axes of the graphs. We based this representation on the method developed in Bana e Costa et al. (2016).

For the input-oriented case, we must replace the linearization constraint in (10) with a constraint that states that the sum of the input multipliers should be one. This process is similar to the one described by Bana and Costa et al. (2016) for the single-stage DEA model. Therefore, your elaboration mode is identical in this case in relation to obtaining the modified weights, the efficiencies and the modified virtual inputs and the modified virtual outputs.

The logic presented above is valid for the overall stage and for the first stage, because in these cases, linearization interferes with the elaboration of the graph, causing all DMUs to be represented on the same line. However, it is possible to note that for the second stage, the efficiency is independent of the modified exogenous virtual input. So, we do not need to modify the obtained weights in model (10), because the linearization of the model does not cause problems for the representation.

To represent the frontier of efficiency in this stage, we simply should plot the DMUs in a graph where the x-axis is the linear combination that plays the role of virtual input for that stage, while the y-axis is the linear combination that plays the role of virtual output for that same stage.

Then, for the first stage, the modified exogenous virtual input plays the role of virtual input (x-axis) and the modified virtual intermediate measure plays the role of the virtual output (y-axis). For the second stage, as we don't need to modify the weights, the virtual intermediate measure plays the role of virtual input (x-axis) and the exogenous virtual output plays the role of the virtual output (y-axis). For the overall process, the modified exogenous virtual input plays the role of the virtual input (x-axis) and the modified exogenous virtual output plays the role of virtual output (y-axis).

Finally, the model leads to the efficient frontier becoming a 45-degree line, which starts from the origin and bisects the first quadrant of the graph, as seen in Bana e Costa et al. (2016).

We can also obtain a two-dimensional representation graph for the output-oriented case. In that case, the exogenous virtual output is always equal to 1, the only difference is that with this orientation, the linearization of the model does not interfere with the two-dimensional representation of the first stage. So, for the first stage, we do not need to modify the weights obtained by the model.

With this representation we can identify the efficients and inefficients DMUs. Also, similar to the one seen in Figure 4, we can draw horizontal lines from the DMU under analysis to calculate the efficiency of that DMU. In addition, we can observe whether the DMUs are closer to the efficient frontier in the first or second stage. Finally, a decision-maker can observe which efficient DMUs are closest to their DMU, and thus use them as benchmarks to improve their DMU processes.

We could propose a model where both representations of internal stages would be on the same graph, similar to that presented in Kao & Hwang (2014). In this case, the representation of the first stage would be in the first quadrant and the representation of the second stage in the second quadrant. However, for this to be possible we should have modified the weight of both stages, and in this model, we prefer to perform the modification only when it is extremely necessary.

4 CASE STUDY

We have applied our approaches in the study presented in Kao & Hwang (2008), which evaluated 24 Taiwanese non-life insurance companies. The overall process was divided in two sub-processes: premium acquisition and profit generation.

The inputs used were Operation expenses (x_{1j}) and Insurance expenses (x_{2j}), the intermediate variables used were Underwriting profit (z_{1j}) and Investment profit (z_{2j}) while the outputs used were Direct written premiums (y_{1j}) and Reinsurance premiums (y_{2j}). All intermediate variables are first stage outputs and second stage inputs.

As pointed out by Kao & Hwang (2008), the first stage measures the performance in marketing the insurance service, while second stage measures performance in generating profit from insurance premiums.

4.1 The Despotis et al. (2016a) model

First, we have obtained the efficiencies at each stage, as well as the virtual inputs and virtual outputs that are used to represent the two-dimensional efficient frontier. As this model provides unique weights for the DMUs, we could use the values obtained. These results are depicted in Table 3.

Table 3 – Virtual input, virtual output, and efficiencies of each stage in the Despotis et al. (2016a) model.

DMU	Virtual input	Virtual output	E_0^1	E_0^2	$E_0^1 * E_0^2 = E_0$
1	1.015435	0.7054	0.9848	0.7054	0.6947
2	1.002908	0.626	0.9971	0.626	0.6242
3	1.449275	1	0.69	1	0.69
4	1.392564	0.4202	0.7181	0.4202	0.3018
5	1.248284	0.9457	0.8011	0.9457	0.7577
6	1.039609	0.4037	0.9619	0.4037	0.3883
7	1.464772	0.4026	0.6827	0.4026	0.2748
8	1.481921	0.4076	0.6748	0.4076	0.275
9	1.059659	0.2323	0.9437	0.2323	0.2192
10	1.274697	0.5597	0.7845	0.5597	0.4391
11	1.449485	0.2276	0.6899	0.2276	0.157
12	1	0.7596	1	0.7596	0.7596
13	1.471887	0.3052	0.6794	0.3052	0.2073
14	1.475579	0.4222	0.6777	0.4222	0.2861
15	1.067122	0.6376	0.9371	0.6376	0.5976
16	1.127269	0.3597	0.8871	0.3597	0.3191
17	1.764291	0.6183	0.5668	0.6183	0.3504
18	1.300221	0.3335	0.7691	0.3335	0.2565
19	1.003814	0.412	0.9962	0.412	0.4104
20	1.29668	0.6763	0.7712	0.6763	0.5216
21	1.345171	0.2668	0.7434	0.2668	0.1984
22	1.696353	1	0.5895	1	0.5895
23	1.22835	0.5079	0.8141	0.5079	0.4135
24	1.209629	0.1255	0.8267	0.1255	0.1037

Figure 7 shows the two-dimensional representation for the proposed model. We can see that no DMU is represented on the efficient frontier as the results in Table 3 pointed out. The most efficient DMU is DMU 12, which is the nearest DMU to the efficient frontier.

In the two-dimensional representation, we can see that DMU 12 is efficient in the first stage, and it is plotted in $\overline{AB}$, where the virtual input is equal to 1. DMUs 2 and 19 also seem to be efficient in the first stage due to their proximity to the efficient frontier, however, they have values very close to 1, hence this location in the graph. Efficient DMUs in the second stage are plotted in $\overline{BC}$ (DMUs 3 and 22), where the virtual output is equal to 1.

The function $\frac{1}{I} = O$ generates the hyperbola $\overline{BD}$ that divides the graph into two regions. DMUs 3, 5, 17, and 22, which are represented above the hyperbola, are more efficient in the second than the first stage, while the other DMUs, represented below the hyperbola, are more efficient in the first stage. If a DMU were represented on the hyperbola, it would have the same efficiency in both stages, which did not happen in this study. This scenario indicates that DMUs generally perform better in the first stage than in the second.

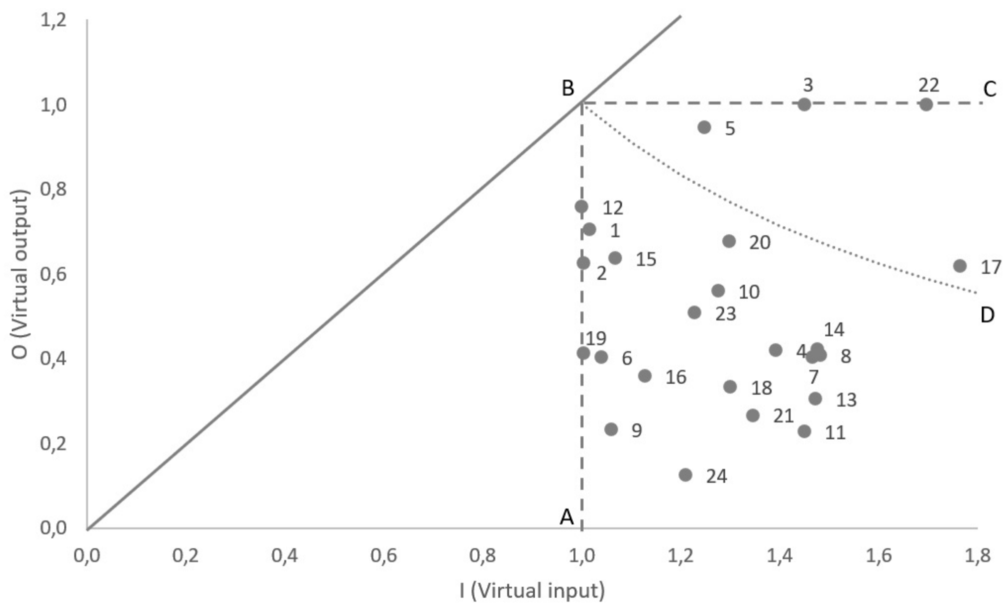

Figure 7 – Two-dimensional frontier representation in the Despotis et al. (2016a) model.

This representation has the the advantage of providing information from more than one stage in the same representation. So, a decision-maker with access to Figure 7 can have an overview of each DMUs' overall efficiency, as well as each DMUs' internal stage efficiency.

4.2 The relational multiplicative NDEA model

As in the original study, we have oriented this model to inputs. This problem has a unique set of optimal weights, which does not always occur.

We present the efficiencies indexes and the linear combinations used as axes of each stage in Table 4.

Table 4 – Virtual inputs, virtual outputs, and efficiencies of each stage for the relational multiplicative model.

U	First Stage			Second Stage			Overall Process		
	Virtual input	Virtual output	E_0^1	Virtual input	Virtual output	E_0^2	Virtual input	Virtual output	E_0
1	803991.1	798021.2	0.993	0.992	0.699	0.704	803991.1	562178.4	0.699
2	1381822	1381822	0.998	1	0.625	0.626	1381822	864139.7	0.625
3	743793.2	513232.3	0.690	0.690	0.690	1	743793.2	513232.3	0.690
4	601320	437820.3	0.724	0.728	0.305	0.420	601320	183736.6	0.304
5	4349626	3613079	0.830	0.830	0.766	0.923	4349626	3336088	0.767
6	698879.6	671358.7	0.961	0.960	0.389	0.406	698879.6	272341.1	0.390
7	1572159	1054353	0.671	0.670	0.276	0.412	1572159	434822.1	0.277
8	1873530	1048827	0.663	0.559	0.276	0.415	1873530	518226.3	0.275
9	1109857	1109857	1	1	0.223	0.223	1109857	247814.3	0.223
10	1303249	1132234	0.862	0.868	0.467	0.541	1303249	609185.7	0.466
11	672414	434913.6	0.647	0.646	0.163	0.253	672414	110224.7	0.164
12	650952	650952	1	1	0.759	0.760	650952	494450.4	0.760
13	1368802	816983.9	0.672	0.596	0.215	0.309	1368802	295102.4	0.208
14	1094028	732913	0.670	0.669	0.288	0.431	1094028	315779.5	0.289
15	771871.8	771871.8	1	1	0.613	0.614	771871.8	473797.4	0.614
16	477814.9	415065.5	0.886	0.868	0.332	0.361	477814.9	158850.6	0.320
17	1180258	740736.7	0.628	0.627	0.360	0.574	1180258	424906.9	0.360
18	602106.2	477796	0.794	0.793	0.258	0.326	602106.2	155851.8	0.259
19	159422	159422	1	1	0.411	0.411	159422	65553.82	0.411
20	53518	49943.93	0.934	0.933	0.546	0.586	53518	29250.05	0.547
21	30787.92	22539.84	0.732	0.732	0.200	0.274	30787.92	6181.63	0.200
22	10934.47	6446.133	0.589	0.589	0.589	1	10934.47	6446.133	0.589
23	35196.25	29655.01	0.843	0.842	0.420	0.499	35196.25	14794.56	0.420
24	163297	70003.68	0.429	0.428	0.134	0.314	163297	22014.27	0.135

Figure 8a shows the two-dimensional representation of the efficient frontier for the first stage. We can note that we have plotted efficient DMUs at the efficient frontier. DMU 2, which is not efficient, looks efficient on the graph, but this happens because it has high efficiency of 0.998. Also, we have represented many DMUs near the (0,0). For this reason, we have zoomed the circled region in Figure 8b. A decision-maker that has access to both figures can have an overall perception of the efficiency distribution of DMUs in the first stage.

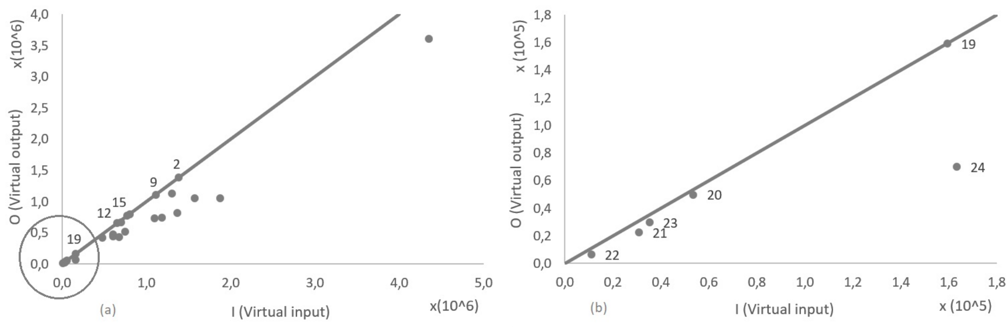

Figure 8 – Two-dimensional frontier representation for the first stage of the relational multiplicative model.

Graphically, it is possible to observe that DMUs 9, 12, 15 and 19 are efficient in the analyzed stage. Furthermore, in general, DMUs are close to the efficient frontier, which indicates that DMUs perform well at this stage. It is possible, as pointed out before, to obtain the projections of virtual inputs for each DMU. It can also be interesting to observe the DMUs that are close together in the graph to see if they have similarities in terms of their production process.

Similarly, we have developed the two-dimensional efficient frontier for the second stage, as shown in Figure 9. Comparing this graph with the graph of the first stage, we can note that efficiencies are lower in this stage, which indicates that DMUs generally perform worse at this stage. Also, DMUs 3 and 22 are efficient and are plotted on the frontier and DMU 5 has a satisfactory performance.

Finally, we have represented the efficient frontier for the overall process in Figure 10a. In the overall process, no DMU is efficient. Consequently, we have not represented any DMU at the efficient frontier. As seen in stage one, some DMUs are near the (0,0), then another graph was constructed for these DMUs, as shown in Figure 10b.

Graphically, it is possible to see that the overall process has similarities with the second stage. Therefore, it is recommended that an analysis be carried out to see why these similarities occur.

All two-dimensional graphical frontier representation follows the same properties. In all of them, it is possible to check if a DMU is efficient and it is also possible to find the efficiency of each DMU just by looking at its axes.

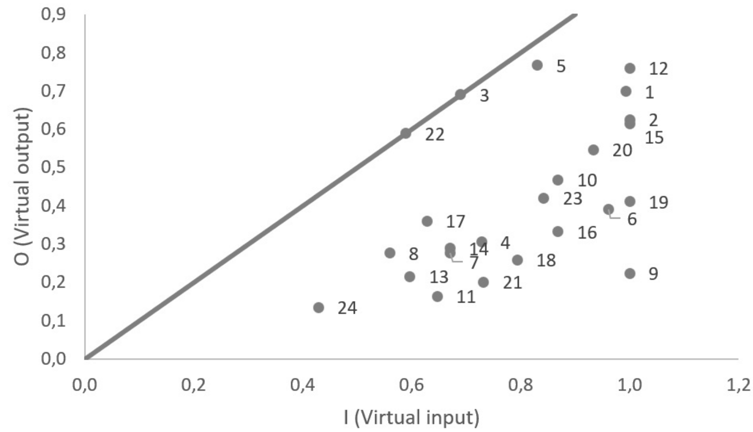

Figure 9 – Two-dimensional frontier representation for the second stage of the relational multiplicative model.

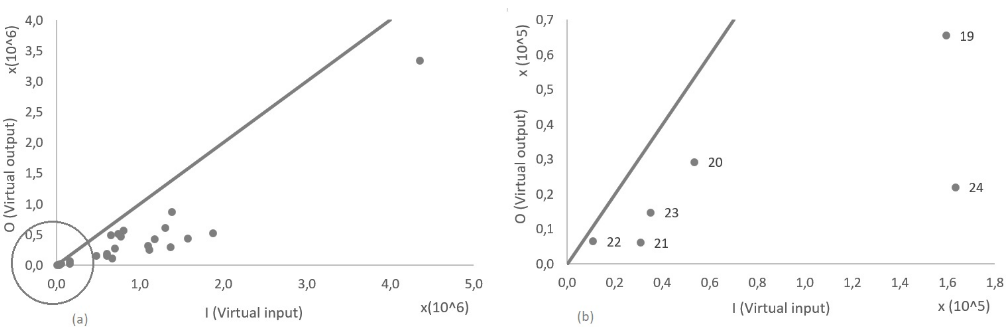

Figure 10 – Two-dimensional frontier representation for the overall process of the relational multiplicative model.

5 CONCLUSIONS

In this paper, we have introduced an approach to provide a two-dimensional representation of the efficient frontier for some Network DEA models with multiple variables. The models analyzed were the relational multiplicative model (Kao & Hwang, 2008) and the model developed in Despotis et al. (2016a) with the multiplicative aggregation. In both models, it was possible to obtain the two-dimensional representation of the efficient frontier for all DMUs.

In the relational multiplicative model, we have proposed one graph for the overall process and one graph for each internal stage. In the Despotis et al. (2016a) model, we have proposed only one graph for the overall process. However, this graph retains information not only about the overall process but also for the first and second stages. Another advantage of this proposal is that we do not need to modify the values of the virtual inputs and virtual outputs for the development of the graph.

As well as in Bana e Costa et al. (2016), in our approach, the efficient frontier is clearly defined, as is the DMUs' location to this frontier. However, our main contribution, compared to the first paper, is to develop a two-dimensional representation model to be applied in Network DEA simply and coherently. These two-dimensional representations may be valuable tools to help managers not familiar with linear programming to make decisions based on the results found in NDEA models. Also, in the two-dimensional representation developed for the model proposed in Despotis et al. (2016a), we have the advantage of providing information from the overall process and each internal stage in the same graphic. Thus, we add more information and more details about DMUs.

As suggestions for future studies, we propose the analysis of the multiple optimal weights for the relational model and its representations in the graph. Besides that, we propose extending the model presented for different types of return to scale (Benicio and Soares de Mello, 2019) and other Network DEA models, as the additive model (Chen et al., 2009), the relational model (Kao, 2009), multi-stage processes (Despotis et al., 2016b; Zhang et al., 2019; Guo et al., 2017) and other models (Tone and Tsutsui, 2009; Guo and Zhu, 2017; Wang et al., 2019). Another suggestion for future studies is to verify how to determine the target for inefficient DMUs at the efficient frontier, for that, it can be applied the method proposed in Chen et al. (2010).

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