SIMPLE AND WEAK Δ-INVARIANT POLYHEDRAL SETS FOR DISCRETE-TIME SINGULAR SYSTEMS

Eugênio B. Castelan*
eugenio@das.ufsc.br

Sophie Tarbouriech†
tarbour@laas.fr

*DAS / CTC / UFSC, Departamento de Automação e Sistemas, 88.040-900 - Florianópolis (S.C.), Brazil

†LAAS - CNRS, 7, Avenue du Colonel Roche, 31077 – Toulouse Cedex 4, France

ABSTRACT

In this paper, necessary and sufficient conditions for the positive invariance of convex polyhedra with respect to linear discrete-time singular systems subject to bounded additive disturbances are established. New notions of Δ-invariance under different assumptions on the initial conditions are defined. Specifically, the notions of *simple* and *weak* Δ-invariance are considered. They can be seen as extensions of the Δ-positive invariance concept used for the regular linear systems with additive disturbances. The results are presented by considering classical equivalent system representations for linear singular systems.

KEYWORDS: Singular systems, Disturbances, Convex polyhedra, Invariance, Initial conditions.

RESUMO

Apresentam-se condições necessárias e suficientes para a invariância positiva de poliédros convexos relativamente a um sistema singular, linear e em tempo discreto, sujeito a perturbações aditivas e limitadas. Introduz-se as noções de Δ-invariância *simples* e de Δ-invariância *fraca*, associadas a diferentes hipóteses a serem verificadas pelas condições iniciais. Estas noções podem ser consideradas como extensões, para o caso de sistemas singulares, do conceito de Δ-invariância utilizado no caso de sistemas regulares sujeitos a perturbações. Os resultados são desenvolvidos a partir de duas formas usuais de representação de sistemas singulares lineares.

PALAVRAS-CHAVE: Sistemas singulares, Perturbações, Poliédros convexos, Invariância, Condições iniciais.

1 INTRODUCTION

The use of the positive invariance property in the control of constrained dynamical systems has been receiving much attention in the last years (Blanchini, 1990; Blanchini, 1994; De Santis, 1994; Georgiou and Krikelis, 1991; Gilbert and Tan, 1991; Hennet, 1989; Hennet and Béziat, 1991; Kolmanovski and Gilbert, 1995; Milani and Dórea, 1996; Tarbouriech and Gomes da Silva Jr., 1997; Tarbouriech and Castelan, 1993; Tarbouriech and Castelan, 1995). This property is used, for instance, to guarantee the maintenance of the state trajectories of a controlled system in the interior of some prescribed sets of admissible states determined from some sets of control or state constraints. External disturbances and/or parametric perturbations can also be considered. The positive invariance in presence of disturbances is commonly referred in the literature as Δ-invariance (Blanchini, 1990; De Santis, 1994; Kolmanovski and Gilbert, 1995).

However, few works exist dealing with the positive invariance property in the case of linear singular systems (Georgiou and Krikelis, 1991; Tarbouriech and Castelan, 1993; Tarbouriech and Castelan, 1995). Furthermore, these works do not consider the presence of external disturbances

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in the considered model.

The objective of this paper is to present necessary and sufficient algebraic conditions to guarantee the positive invariance property of convex polyhedra with respect to singular discrete-time systems subject to additive disturbances belonging to a convex set Δ. The general notion of Δ-invariance of a domain $\mathcal{D} \subset \Re^n$ with respect to a dynamical system is associated to the maintenance of the system trajectories within the domain $\mathcal{D}$ for any initial condition belonging to $\mathcal{D}$ and for any sequence of admissible disturbances belonging to Δ. Thus, a major objective of this paper consists in extending the Δ-invariance concept used for the regular linear systems to the linear singular systems.

Due to the specificities of singular systems in terms of initial conditions (initial conditions may be consistent or not), some different notions have to be considered. Hence, from the assumptions on the initial conditions with respect to the domain $\mathcal{D}$, we define the notions of *simple* and *weak* Δ-invariance.

Considering convex polyhedra, the algebraic characterization of both the *simple* and *weak* Δ-invariance properties are obtained for an equivalent representation of special interest in the theory of linear singular systems, called Differential-Algebraic Form.

The paper is organized as follows. The problem to be treated and the *simple* and *weak* Δ-invariance properties are formulated in section 2. Section 3 presents the main results in terms of the Differential-Algebraic Representation. Some general comments about the proposed results are given in section 4. An illustrative example in section 5 allows to show an application of the results to a constrained control problem. Section 6 ends the paper with some concluding remarks.

2 PROBLEM PRESENTATION

Consider a linear discrete-time singular perturbed system described by:

$$Ex_{k+1} = A_0 x_k + D w_k \tag{1}$$

where $E \in \Re^{n\times n}$ with $rank(E) = q \leq n$, $A_0 \in \Re^{n\times n}$, $D \in \Re^{n\times d}$. $x_k \in \Re^n$ and $w_k \in \Re^d$ represent respectively the state and additive disturbance vectors. System (1) can, for instance, represent the closed-loop dynamic behavior of a linear singular system

$$Ex_{k+1} = Ax_k + Bu_k + Dw_k \tag{2}$$

controlled by a static full state feedback or static output feedback. In these cases, we have $A_0 = A + BF$ or $A_0 = A + BKC$, respectively.

Since for control purposes system (1) represents controlled systems, we shall consider that it has a unique solution for all initial condition $x_0 \in \Re^n$ and for any admissible sequence $\{w_k\}$, with $w_k \in \Re^d$, $\forall k \geq 0$, and also that the system is causal. Thus, the following assumption is supposed to hold throughout this note (for details see, for instance, Dai (1989) and Lewis (1986)).

Assumption 1 The pair (E, A_0) is assumed to be regular and system (1) is supposed to be impulse free.

However, the dynamic behavior of system (1) can present discontinuous behavior at $k = 0$ if the associated initial condition x_0 is not consistent. For any given initial condition x_0, the actual state at $k = 0$ is denoted $x_k|_{k=0} = x_{0^+}$. The *set of consistent initial conditions* (Dai, 1989) is the set of x_0 which prevents the system of discontinuous behavior:

$$\mathcal{I}_0 = \{x_0 \in \Re^n\, ;\, x_{0^+} = x_0\} \tag{3}$$

It is well-known that in the case of the unperturbed system ($w_k = 0, \forall k$), $\mathcal{I}_0$ corresponds to the subspace of $\Re^n$ spanned by the finite eigenvectors of the pair (E, A_0). But, in the case of system (1), $\mathcal{I}_0$ depends also on w_0. In general, if $w_0 \neq 0$, a finite jump with amplitude $|x_0 - x_{0^+}|$ may occur at $k = 0$ for any initial condition x_0.

Thus, let us now introduce different notions of Δ-invariance depending on certain assumptions made about the initial conditions. The first definition is the closest to the practical situation of hard constraints.

Definition 1 A nonempty set $\mathcal{D} \subset \Re^n$ is a simple Δ-invariant domain with respect to system (1) if for any initial condition $x_0 \in \mathcal{D}$ and sequence $\{w_k\}$, with $w_k \in \Delta$, $\forall k \geq 0$, it follows that $x_{0^+} \in \mathcal{D}$ and $x_k \in \mathcal{D}$, $\forall k \geq 1$.

The second definition is weaker in terms of hypothesis because it assumes that only the initial condition x_0 belongs to the invariant domain. It may be used, for instance, in practical situations of soft constraints or, as indicated by the comments in section 4, for stability and disturbance rejection purposes.

Definition 2 A nonempty set $\mathcal{D} \subset \Re^n$ is a weak Δ-invariant domain with respect to system (1) if for any initial condition $x_0 \in \mathcal{D}$ and sequence $\{w_k\}$, with $w_k \in \Delta$, $\forall k \geq 0$, it follows that $x_k \in \mathcal{D}$, $\forall k \geq 1$.

Other two notions of Δ-invariance have been considered in Castelan and Tarbouriech (1996) and Castelan and Tarbouriech (2000). They are omitted in the present paper since the notions of *simple* and *weak* Δ-invariance better cope with realistic practical situations.

In this work, we are mainly concerned with the application of Definitions 1 and 2 to the case of closed polyhedral domains. In practical control problems, polyhedral domains can represent linear constraints on the state and limits for the allowed disturbances. The polyhedral sets of states and disturbances to be considered in the sequel are defined by:

$$R(G,\rho)=\{x\in\Re^n\ ;\ Gx\leq\rho\}\ , G\in\Re^{g\times n},\ \rho\in\Re^g \tag{4}$$

and

$$\Delta=R(T,\mu)=\{w\in\Re^d\ ;\ Tw\leq\mu\}\ , T\in\Re^{p\times d},\ \mu\in\Re^p \tag{5}$$

Any nonempty convex polyhedron of $\Re^n$ or $\Re^p$ can be characterized by (4) or (5), respectively. By convention the inequalities between vectors are component-wise.

Thus, the primary objective of this work is to give algebraic necessary and sufficient conditions for both the *simple* and *weak* $R(T,\mu)$-invariance of convex polyhedron $R(G,\rho)$ with respect to system (1). To accomplish the stated objectives, we shall consider an equivalent representation of system (1), called *Differential-Algebraic form,* and the corresponding representation of the polyhedral set $R(G,\rho)$. Another particular equivalent representation of system (1), under a *Standard form,* is also used in the proofs of the proposed results.

In general, an equivalent representation of system (1) can be obtained as follows (see Dai (1989)). Let $\tilde{Q}$ and $\tilde{P}$ be two nonsingular n-order matrices. Then, by considering the change of coordinates $x=\tilde{P}\tilde{x}$, the following equivalent representation of system (1) can be defined:

$$\tilde{E}\tilde{x}_{k+1}=\tilde{A}_0\tilde{x}_k+\tilde{D}w_k \tag{6}$$

where: $\tilde{E}=\tilde{Q}E\tilde{P}$, $\tilde{A}_0=\tilde{Q}A_0\tilde{P}$, and $\tilde{D}=\tilde{Q}D$. The corresponding representation of the polyhedral set $R(G,\rho)$ is given by: $R(\tilde{G},\rho)=\{\tilde{x}\in\Re^n\ ;\ \tilde{G}\tilde{x}\leq\rho\}$, with $\tilde{G}=G\tilde{P}$.

3 MAIN RESULTS

Since $rank(E)=q$, there exist nonsingular n-order matrices $Q=\begin{bmatrix} Q_1' \\ Q_2' \end{bmatrix}$ and $P=\begin{bmatrix} P_1 & P_2 \end{bmatrix}$ such that $QEP=\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}$ (Dai, 1989). Thus, by considering the change of coordinates $x=\begin{bmatrix} P_1 & P_2 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}$, with $x^1\in\Re^q$ and $x^2\in\Re^{(n-q)}$, system (1) can be rewritten as

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x^1_{k+1} \\ x^2_{k+1} \end{bmatrix}=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x^1_k \\ x^2_k \end{bmatrix}+\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}w_k \tag{7}$$

where: $QA_0P=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$, $QD=\begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$, with

$$A_1\in\Re^{q\times q}\ ,\ A_2\in\Re^{q\times(n-q)}\ ,\ A_3\in\Re^{(n-q)\times q}\ ,$$
$$A_4\in\Re^{(n-q)\times(n-q)}\ ,\ D_1\in\Re^{q\times d}\ ,\ D_2\in\Re^{(n-q)\times d}\ .$$

Since the singular system is supposed to be impulse free, matrix A_4 is nonsingular (Dai, 1989; Lewis, 1986). The corresponding representation of the polyhedron $R(G,\rho)$ is given by

$$R(G_1,G_2,\rho)=\left\{\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}\in\Re^n\ ;\ \begin{bmatrix} G_1\, G_2 \end{bmatrix}\begin{bmatrix} x^1 \\ x^2 \end{bmatrix}\leq\rho\right\} \tag{8}$$

where: $G_1=GP_1\in\Re^{g\times q}$ and $G_2=GP_2\in\Re^{g\times(n-q)}$.

The equivalent representation (7) gives a good meaning to singular systems: the system is composed of dynamic subsystems and an algebraic part which represents the connection between subsystems. Thus, descriptions of dynamical systems under the Differential-Algebraic Form arise naturally when systems are formed from interconnected systems (Dai, 1989). Otherwise, a representation of any linear singular system under the form (7) can be generally obtained from a *singular value decomposition* of matrix E (Dai, 1989).

The following result presents necessary and sufficient algebraic conditions for the *simple* Δ-invariance property of convex polyhedra.

Proposition 1 The polyhedral set $R(G_1,G_2,\rho)$ is simply $R(T,\mu)$-invariant with respect to system (7) if and only if there exist nonnegative matrices $S_1\in\Re^{g\times g}$, $S_2\in\Re^{g\times p}$, $S_3\in\Re^{g\times p}$, $S_4\in\Re^{g\times g}$, $S_5\in\Re^{g\times p}$ and $S_6\in\Re^{g\times p}$ such that:

$$\begin{aligned}
S_1\bar{G}_1 &= \bar{G}_1(A_1-A_2A_4^{-1}A_3) && (9)\\
S_1\bar{G}_2 &= 0 && (10)\\
S_2T &= \bar{G}_1(D_1-A_2A_4^{-1}D_2) && (11)\\
S_3T &= -\bar{G}_2A_4^{-1}D_2 && (12)\\
S_4\bar{G}_1 &= \bar{G}_1 && (13)\\
S_4\bar{G}_2 &= 0 && (14)\\
S_5T &= -\bar{G}_2A_4^{-1}D_2 && (15)\\
S_6T &= 0 && (16)\\
S_1\rho+(S_2+S_3)\mu &\leq \rho && (17)\\
S_4\rho+(S_5+S_6)\mu &\leq \rho && (18)
\end{aligned}$$

where, by definition: $\bar{G}_1=G_1-G_2A_4^{-1}A_3$ and $\bar{G}_2=G_2$.

Proof: To develop the proof, we shall represent the system under a *standard form*. This equivalent representation decomposes the system into *slow* and *fast* subsystems related, respectively, to the finite and infinite eigenvalues of the (impulse free) singular system (Dai, 1989). Thus, by considering the nonsingular n-order matrices $\bar{Q} = \begin{bmatrix} I_q & -A_2A_4^{-1} \\ 0 & A_4^{-1} \end{bmatrix}$, $\bar{P} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \end{bmatrix} = \begin{bmatrix} I_q & 0 \\ -A_4^{-1}A_3 & I_{n-q} \end{bmatrix}$ and the change of coordinates $\begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} \bar{P}_1 & \bar{P}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}$, system (7) can be rewritten as

$$\begin{bmatrix} I_q & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}^1_{k+1} \\ \bar{x}^2_{k+1} \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & 0 \\ 0 & I_{n-q} \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \end{bmatrix} + \begin{bmatrix} \bar{D}_1 \\ \bar{D}_2 \end{bmatrix} w_k \tag{19}$$

where:
$\bar{A}_1 = (A_1 - A_2A_4^{-1}A_3)$, $\bar{D}_1 = (D_1 - A_2A_4^{-1}D_2)$ and $\bar{D}_2 = A_4^{-1}D_2$.

The corresponding representation of the polyhedron $R(G, \rho)$ is given by

$$R(\bar{G}_1, \bar{G}_2, \rho) = \left\{ \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} \in \Re^n \, ; \, \begin{bmatrix} \bar{G}_1 \; \bar{G}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix} \leq \rho \right\} \tag{20}$$

Notice that with respect to system (19), we always have $\bar{x}^2_k = -\bar{D}_2 w_k$, $\forall k$. In particular, if $k = 0$ and an initial condition $\bar{x}_0 = \begin{bmatrix} \bar{x}^1_0 \\ \bar{x}^2_0 \end{bmatrix}$ is considered, it follows that the actual substate $\bar{x}^2_k|_{k=0} = -\bar{D}_2 w_0$ and, hence, $\bar{x}_{0^+} = \begin{bmatrix} \bar{x}^1_0 \\ -\bar{D}_2 w_0 \end{bmatrix}$. Thus, in general a jump occurs at $k = 0$, which is now considered.

>From Definition 1, the *simple* $R(T, \mu)$-invariance of $R(\bar{G}_1, \bar{G}_2, \rho)$ with respect to system (19) corresponds to:

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^1_{k+1} \\ \bar{x}^2_{k+1} \end{bmatrix} \leq \rho$$

and

$$\begin{bmatrix} \bar{G}_1 & 0 & -\bar{G}_2\bar{D}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \\ w_k \end{bmatrix} \leq \rho$$

for all $\begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \end{bmatrix}$ and w_k such that

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 \\ 0 & 0 & T \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \\ w_k \end{bmatrix} \leq \begin{bmatrix} \rho \\ \mu \end{bmatrix} \tag{21}$$

From (19), this condition also writes:

$$\begin{bmatrix} \bar{G}_1\bar{A}_1 & 0 & \bar{G}_1\bar{D}_1 & -\bar{G}_2\bar{D}_2 \\ \bar{G}_1 & 0 & -\bar{G}_2\bar{D}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \\ w_k \\ w_{k+1} \end{bmatrix} \leq \begin{bmatrix} \rho \\ \rho \end{bmatrix}$$

for all $\begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \end{bmatrix}$, w_k and w_{k+1} such that

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \\ w_k \\ w_{k+1} \end{bmatrix} \leq \begin{bmatrix} \rho \\ \mu \\ \mu \end{bmatrix}.$$

By using the extended Farkas' lemma (Hennet, 1989), it follows that a necessary and sufficient condition for the *weak* Δ-invariance of $R(\bar{G}_1, \bar{G}_2, \rho)$ is the existence of a nonnegative matrix $\begin{bmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \end{bmatrix}$ satisfying both:

$$\begin{bmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \end{bmatrix} \begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix} =$$

$$\begin{bmatrix} \bar{G}_1\bar{A}_1 & 0 & \bar{G}_1\bar{D}_1 & -\bar{G}_2\bar{D}_2 \\ \bar{G}_1 & 0 & -\bar{G}_2\bar{D}_2 & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} S_1 & S_2 & S_3 \\ S_4 & S_5 & S_6 \end{bmatrix} \begin{bmatrix} \rho \\ \mu \\ \mu \end{bmatrix} \leq \begin{bmatrix} \rho \\ \rho \end{bmatrix}.$$

Therefore relations (9)-(18) of Proposition 1 follow. □

Relative to the *weak* Δ-invariance property we get the following result.

Proposition 2 The polyhedral set $R(G_1, G_2, \rho)$ is weakly $R(T, \mu)$-invariant with respect to system (7) if and only if there exist nonnegative matrices $W_1 \in \Re^{g \times g}$, $W_2 \in \Re^{g \times p}$ and $W_3 \in \Re^{g \times p}$ such that

$$W_1\bar{G}_1 = \bar{G}_1(A_1 - A_2A_4^{-1}A_3) \tag{22}$$
$$W_1\bar{G}_2 = 0 \tag{23}$$
$$W_2T = \bar{G}_1(D_1 - A_2A_4^{-1}D_2) \tag{24}$$
$$W_3T = -\bar{G}_2A_4^{-1}D_2 \tag{25}$$
$$W_1\rho + (W_2 + W_3)\mu \leq \rho \tag{26}$$

where, by definition: $\bar{G}_1 = G_1 - G_2A_4^{-1}A_3$ and $\bar{G}_2 = G_2$.

Proof: As in the previous proof, the standard form (19) is used to obtain the desired invariance relations. From Definition 2, a necessary and sufficient condition for the *weak* $R(T, \mu)$-invariance of $R(\bar{G}_1, \bar{G}_2, \rho)$ with respect to system (19) is

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 \end{bmatrix} \begin{bmatrix} \bar{x}^1_{k+1} \\ \bar{x}^2_{k+1} \end{bmatrix} \leq \rho \tag{27}$$

for all $\begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \end{bmatrix}$ and w_k such that

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 \\ 0 & 0 & T \end{bmatrix} \begin{bmatrix} \bar{x}^1_k \\ \bar{x}^2_k \\ w_k \end{bmatrix} \leq \begin{bmatrix} \rho \\ \mu \end{bmatrix}. \tag{28}$$

For all $\bar{x}_k \in R(\bar{G}_1, \bar{G}_2, \rho)$ and for all admissible disturbances $w_k \in R(T, \mu)$, $\forall k$, one can also write:

$$\begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix} \begin{bmatrix} \bar{x}_k^1 \\ \bar{x}_k^2 \\ w_k \\ w_{k+1} \end{bmatrix} \leq \begin{bmatrix} \rho \\ \mu \\ \mu \end{bmatrix} \qquad (29)$$

In the same way, since from (19) we get $\bar{x}_{k+1}^2 = -\bar{D}_2 w_{k+1}$, (27) is equivalent to

$$\begin{bmatrix} \bar{G}_1 \bar{A}_1 & 0 & \bar{G}_1 \bar{D}_1 & -\bar{G}_2 \bar{D}_2 \end{bmatrix} \begin{bmatrix} \bar{x}_k^1 \\ \bar{x}_k^2 \\ w_k \\ w_{k+1} \end{bmatrix} \leq \rho \qquad (30)$$

Hence the *weak* $R(T, \mu)$-invariance of $R(\bar{G}_1, \bar{G}_2, \rho)$, expressed by (27) and (28), is obtained when every solution of (29) is also solution of (30), which corresponds to the inclusion of a polyhedral convex set into another polyhedral convex set. Then by applying the extended Farkas' lemma (Hennet, 1989), it follows that a necessary and sufficient condition for the *weak* Δ-invariance of $R(\bar{G}_1, \bar{G}_2, \rho)$ is the existence of a nonnegative matrix $\begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix}$ satisfying both:

$$\begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \begin{bmatrix} \bar{G}_1 & \bar{G}_2 & 0 & 0 \\ 0 & 0 & T & 0 \\ 0 & 0 & 0 & T \end{bmatrix} =$$

$$\begin{bmatrix} \bar{G}_1 \bar{A}_1 & 0 & \bar{G}_1 \bar{D}_1 & -\bar{G}_2 \bar{D}_2 \end{bmatrix}$$

and

$$\begin{bmatrix} W_1 & W_2 & W_3 \end{bmatrix} \begin{bmatrix} \rho \\ \mu \\ \mu \end{bmatrix} \leq \rho \, .$$

Therefore, relations (22)-(26) of Proposition 2 follow. $\square$

We finish this section with the following remarks related to the *Standard form* (19). By recalling the general procedure described in section 2 to obtain an equivalent representation (6), we first remark that (19) can be obtained from (1) by considering $\tilde{Q} = Q\bar{Q}$, $\tilde{P} = P\bar{P}$ and the change of coordinates $x = \tilde{P}\tilde{x}$, where $\tilde{x} = \begin{bmatrix} \bar{x}^1 \\ \bar{x}^2 \end{bmatrix}$. Next, let us recall that the finite eigenvalues of the considered regular and impulse-free singular system are given by the zeros of the characteristic polynomial

$$det\,(\lambda E - A_0) = det\left(\tilde{Q}^{-1} \begin{bmatrix} \lambda I_q - \bar{A}_1 & 0 \\ 0 & I_{n-q} \end{bmatrix} \tilde{P}^{-1}\right).$$

Thus, the q finite eigenvalues of pair (E, A_0) correspond to the eigenvalues of matrix $\bar{A}_1 \in \Re^{q \times q}$ that are the roots of the characteristic equation (Kailath, 1980):

$$det\left(\lambda I_q - (A_1 - A_2 A_4^{-1} A_3)\right) = 0.$$

4 GENERAL COMMENTS

The concept of Δ-invariance reduces to the classical concept of *positive invariance* in the case of unperturbed (regular and impulse free) singular system $Ex_{k+1} = A_0 x_k$. Hence, algebraic characterizations of the positive invariance property of $R(G, \rho)$, as presented by Tarbouriech and Castelan (1993), can be easily obtained from Propositions 1 and 2 by considering $D = 0$, $T = 0$ and $\mu = 0$.

The case of *regular* linear systems can be viewed as a particular case of system (19), by considering only the slow subsystem

$$\bar{x}_{k+1}^1 = \bar{A}_1 \bar{x}_k^1 + \bar{D}_1 w_k \qquad (31)$$

Thus, the classical $R(T, \mu)$-invariance of $R(\bar{G}_1, \rho)$ with respect to system (31) can be characterized by using Proposition 2 with $\bar{G}_2 = 0$, $\bar{D}_2 = 0$ and $W_3 = 0$.

Furthermore, notice that no assumption on the signs of the elements of vectors ρ and μ is considered in the presented results. However, in most control applications the zero-state has to be feasible and we can consider that both vectors ρ and μ are nonnegative, that is, $\rho \in \Re_+^g$ and $\mu \in \Re_+^d$, where $\Re_+^g$ (resp. $\Re_+^d$) denotes the positive orthant of $\Re^g$ (resp. $\Re^d$).

Under the assumption of non-negativeness of ρ and μ, some stability properties of the eigenvalues of matrix W_1 can be deduced from relation (26) by using the theory of nonnegative matrices and M-matrices (Tarbouriech and Castelan, 1993). Thus, relation (22) can be thought as expressing a certain intersection between the spectra of W_1 and $\bar{A}_1$; relation (23) implies the existence of some null eigenvalues in the spectrum of W_1. According to this intersection, some stability properties of the singular system can be deduced since its set of finite eigenvalues is given by the spectrum of $\bar{A}_1$ (Tarbouriech and Castelan, 1993).

As shown by Milani and Dórea (1996), the existence of admissible matrices W_2 and W_3 satisfying relations (24)-(25) is related to the following null-space intersections: $\bar{D}_1\, Ker(T) \subseteq Ker(\bar{G}_1)$ and $-\bar{D}_2\, Ker(T) \subseteq Ker(\bar{G}_2)$.

Also, consider both the set of relations (9)-(18) of Proposition 1 and (22)-(26) of Proposition 2. It is clear that the conditions of Proposition 1 are harder to satisfy than those of Proposition 2. If relations (9)-(18) hold then relations (22)-(26) also hold for $W_1 = S_1$, $W_2 = S_2$, $W_3 = S_3$. Hence, if $R(G, \rho)$ is a *simple* $R(T, \mu)$-invariant domain, then it is also a *weak* $R(T, \mu)$-invariant domain, but the converse is not generally true. Therefore relative to $R(G, \rho)$ the following implication holds:

simple $R(T, \mu)$-invariant $\Rightarrow$ *weak* $R(T, \mu)$-invariant.

This fact follows from the different definitions of every notion. The notion of *weak* Δ-invariance requires that only the initial condition x_0 belongs to $\mathcal{D}$, whereas the notion of *simple* Δ-invariance requires that both the initial condition x_0 and the associated discontinuous state x_{0^+} belong to $\mathcal{D}$. However, a particular case exists where the two studied notions of Δ-invariance become equivalent. This case occurs whenever $R(\bar{G}_1, \bar{G}_2, \rho)$ is unbounded in the directions of sub-state $\bar{x}^2$. In such a case, we necessarily have $G_2 = \bar{G}_2 = 0$ and it means that any jump due to non-consistent initial condition is admitted. Thus, if $R(G_1, \rho) = R(G_1, 0, \rho)$ is a *weak* $R(T, \mu)$-invariant set it is also a *simple* $R(T, \mu)$-invariant set. This can also be verified from the conditions of Propositions 1 and 2.

Since the singular system can describe a kind of interconnected system, its sub-state $\bar{x}^2$ can be considered as a pseudo-state and $\bar{x}_0^2$ represents the interconnection between subsystems at initial time $k = 0$ (Dai, 1989). In this context, non-consistent initial conditions would represent topological changes in the interconnections' level. Therefore, in practical situations, both the choice of matrix $\bar{G}_2 = G_2$ and the used Δ-invariance concept (*simple* and/or *weak*) has to be associated to the possible topological changes in the interconnections between subsystems.

5 ILLUSTRATIVE EXAMPLE

Consider the open-loop discrete-time singular system (2) borrowed from (Tarbouriech and Castelan, 1993), described by the following matrices:

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1.2 & 0 & 0 \\ -1 & -0.7 & -1 \\ 2 & -0.5 & -1.2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \\ 0.5 & 2 \end{bmatrix}, \ D = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{32}$$

The set $R(T, \mu)$ of allowed persistent disturbances is described by

$$T = \begin{bmatrix} 1 \\ -1 \end{bmatrix} ; \ \mu = \gamma \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{33}$$

where γ is some positive scalar.

The vector of admissible controls is constrained to belong to a compact set Ω

$$\Omega = \{u_k \in \Re^m \ ; \ -\varrho \leq u_k \leq \varrho\} \ ; \ \varrho = \begin{bmatrix} \varrho_1 \\ \varrho_2 \end{bmatrix} > \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{34}$$

Thus, assuming that a saturated state-feedback control law is applied

$$u_k = sat(Fx_k) \tag{35}$$

where $sat(u_i) = sign(u_i) \min\{|u_i|, \varrho_i\}$, for $i = 1, 2$, the closed-loop system is given by:

$$Ex_{k+1} = Ax_k + Bsat(Fx_k) + Dw_k \tag{36}$$

This closed-loop system is non-linear but, for any state x_k inside the polyhedral set $S(F, \varrho)$, defined from Ω by (37), the state at $k+1$ is determined by the linear model (2):

$$S(F, \varrho) = \{x_k \in \Re^n \ ; \ -\varrho \leq Fx_k \leq \varrho\} \tag{37}$$

Let us now consider the state feedback matrix $F \in \Re^{2\times 3}$ computed by Tarbouriech and Castelan (1993):

$$F = \left[\begin{array}{c|c} F_1 & F_2 \end{array} \right] = \left[\begin{array}{cc|c} 0 & 1 & 0.9318 \\ -1 & 0 & -0.1164 \end{array} \right] \tag{38}$$

The corresponding closed-loop linear model under the form (1) is determined by matrices E and D given before and by:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} x_{k+1} = \begin{bmatrix} 0.2 & 0.0 & -0.1164 \\ 0.0 & 0.3 & 0.0481 \\ 0.0 & 0.0 & -0.9668 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \omega_k \tag{39}$$

Since the corresponding set $S(F, \varrho)$ is not an invariant domain with respect to (39), saturations may occur for trajectories emanating from $S(F, \varrho)$. Hence, one can be interested in verifying the invariance property in the presence of disturbances from some subset of $S(F, \varrho)$. In this way, we are primarily interested in verifying the $R(T, \mu)$-invariance property of the set $R(G, \rho)$ defined in (4), where G and ρ are given by:

$$G = \begin{bmatrix} 0 & 1 & 0.9318 \\ -1 & 0 & -0.1164 \\ 0 & 0 & 1 \\ 0 & -1 & -0.9318 \\ 1 & 0 & 0.1164 \\ 0 & 0 & -1 \end{bmatrix}, \ \rho = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}.$$

Notice that the considered set $R(G, \rho)$ verifies $R(G, \rho) \subseteq S(F, \varrho)$. In the unperturbed case ($w_k = 0$, $\forall k$), the set $R(G, \rho)$ is a *weak* positively invariant set of system (39). In the perturbed case, the objective is to determine the maximal scalar $\gamma = \gamma_{\max}$ such that $R(G, \rho)$ is *weakly* $R(T, \mu)$-invariant with respect to (39). By using Linear Programming to evaluate the relations of Proposition 2, we obtain $\gamma_{\max} = 0.208815$ with:

$$W_1 = \begin{bmatrix} 0.3000 & 0.0 & 0.0000 & 0.0000 & 0.0 & 0.2795 \\ 0.7632 & 0.2 & 0.0232 & 0.7632 & 0.0 & 0.0000 \\ 0.0000 & 0.0 & 0.0000 & 0.0000 & 0.0 & 0.0000 \\ 0.0000 & 0.0 & 0.2795 & 0.3000 & 0.0 & 0.0000 \\ 0.0000 & 0.0 & 0.0000 & 0.0000 & 0.2 & 0.0232 \\ 0.0000 & 0.0 & 0.0000 & 0.0000 & 0.0 & 0.0000 \end{bmatrix},$$

$$W_2 = \begin{bmatrix} 1.0497 & 0.0000 \\ 0.1203 & 0.0000 \\ 0.0000 & 0.0000 \\ 0.0000 & 1.0497 \\ 0.0000 & 0.1203 \\ 0.0000 & 0.0000 \end{bmatrix}, \; W_3 = \begin{bmatrix} 0.9637 & 0.0000 \\ 0.0000 & 0.1203 \\ 1.0343 & 0.0000 \\ 0.0000 & 0.9637 \\ 0.1203 & 0.0000 \\ 0.0000 & 1.0343 \end{bmatrix}.$$

Let us now consider the polyhedral set $R(\hat{G}, \hat{\rho})$, where:

$$\hat{G} = \begin{bmatrix} F_1 & 0 \\ -F_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \; \hat{\rho} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}.$$

This set corresponds to the intersection $R(G, \rho) \cap \mathcal{I}_0$, where $\mathcal{I}_0$ is the set of consistent initial conditions of system (39) in the unperturbed case. Since both $R(G, \rho)$ and $\mathcal{I}_0$ are invariant domains with respect to (39), the unbounded set $R(\hat{G}, \hat{\rho})$ is also an invariant domain in the disturbance-free case (Tarbouriech and Castelan, 1993). In the perturbed case, it can be verified that $R(\hat{G}, \hat{\rho})$ is both *simply* and *weakly* $R(T, \mu)$-invariant with respect to system (39), for $\gamma = \hat{\gamma}_{\max} = 0.6667$. In particular, the following matrices can be used to verify the relations of Proposition 2:

$$\hat{W}_1 = \begin{bmatrix} 0.3 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.2 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.3 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.2 \end{bmatrix}, \hat{W}_2 = \begin{bmatrix} 1.0497 & 0.0000 \\ 0.1203 & 0.0000 \\ 0.0000 & 1.0497 \\ 0.0000 & 0.1203 \end{bmatrix}$$

and $\hat{W}_3 = 0$.

Let us now show some system trajectories emanating from the above considered Δ-invariant sets. To simplify the analysis, we shall consider that the control bounds, represented by the elements of $\varrho > 0$, are sufficiently large so that no saturations occur, i.e., any considered trajectory evolves inside $S(F, \rho)$ and is determined by the linear model (39).

Fig.1 shows the time-response of the first three components of vector Gx_k (due to symmetry property) by considering the set $R(G, \rho)$ and $\gamma_{\max} = 0.208815$. The considered disturbances are randomly generated so that the corresponding sequence w_k belongs to the interval $[-\gamma_{\max}, \gamma_{\max}]$. By choosing the initial condition $x_0 = \begin{bmatrix} -2.1164 & -1.9318 & 1 \end{bmatrix}'$, one gets $x_{0^+} = \begin{bmatrix} -2.1164 & -1.9318 & 0.2160 \end{bmatrix}'$. Notice that $Gx_0 \leq \rho$ whereas $Gx_{0^+} \not\leq \rho$. But, since $R(G, \rho)$ is a *weak* $R(T, \mu)$-invariant set, one gets $Gx_k \leq \rho$, $\forall k \geq 1$.

Fig.2 shows the time-response of the first two components of vector $\hat{G}x_k$ (due to symmetry property) and the time-response of the last component of vector x_k by considering the set $R(G, \rho) \cap \mathcal{I}_0$ and $\gamma_{\max} = 0.6667$. By choosing the initial condition $x_0 = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix}'$, one gets $x_{0^+} = \begin{bmatrix} 2 & 1 & 0.6896 \end{bmatrix}'$. Notice that both $\hat{G}x_0 \leq \rho$ and $\hat{G}x_{0^+} \leq \rho$.

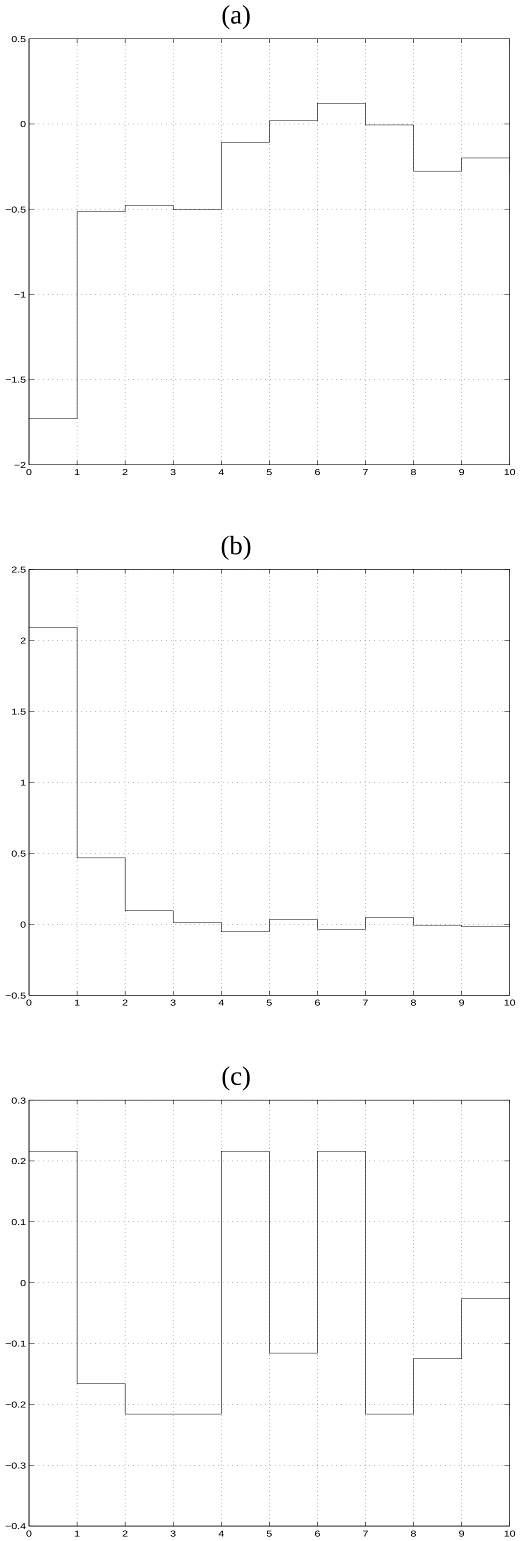

Figure 1: Time-response of the first three components of vector Gx_k

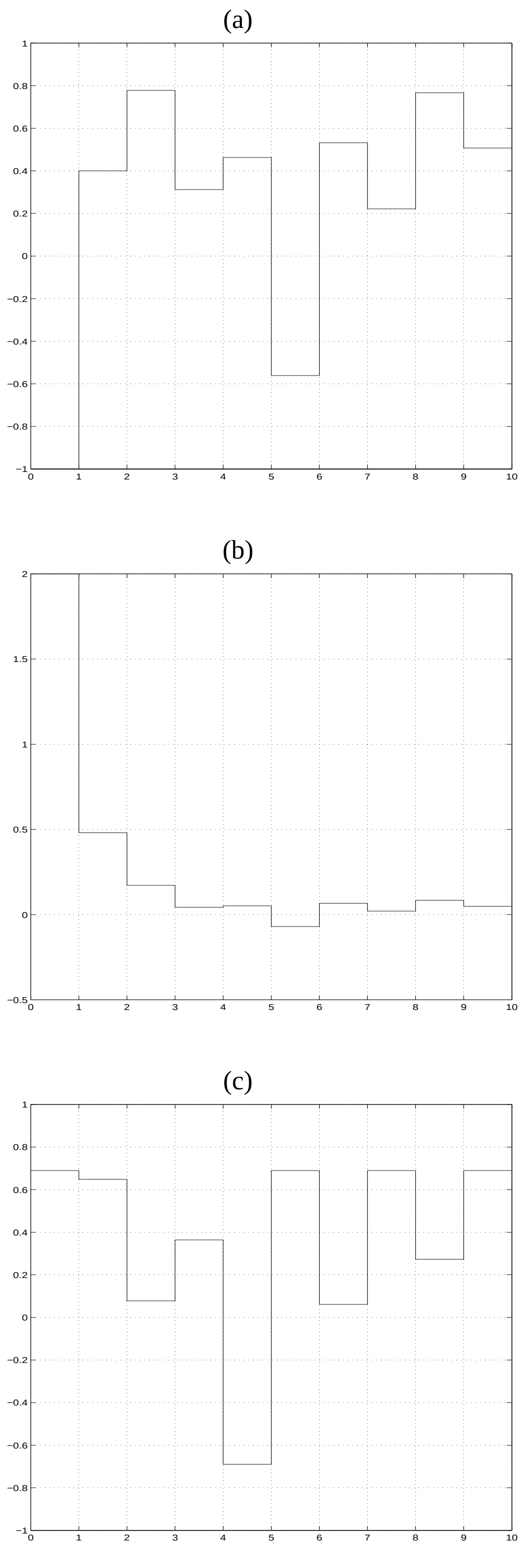

Figure 2: (a)-(b).Time-response of the first two components of $\hat{G}x_k$. (c). Time-response of the last component of x_k

.

6 CONCLUSION

We have presented some results on the invariance property of convex polyhedra with respect to linear singular systems with additive disturbances. In this way, we have defined the notions of *simple* and *weak* Δ-invariance according to the assumptions on the initial conditions. Algebraic necessary and sufficient conditions for characterizing these properties have been proposed relative to two classical equivalent system representations.

As for regular systems, the presented results may be used for solving constrained control problems (Castelan and Tarbouriech, 1996). To this end, the computation of the maximal admissible *weakly* Δ-invariant set contained in a given polyhedron of constraints is considered in Tarbouriech and Castelan (1997). Finally, we remark that a similar technique can be proposed to compute the maximal *simply* Δ-invariant set.

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REFERENCES

Blanchini, F. (1990). Feedback control for linear time-invariant systems with state and control bounds in the presence of disturbances, *IEEE Transactions on Automatic Control* **35**(11): 1231–1234.

Blanchini, F. (1994). Ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions, *IEEE Transactions on Automatic Control* **39**(2): 428–433.

Castelan, E. B. and Tarbouriech, S. (1996). Positively invariant sets for discrete-time singular systems with additive perturbations, *Proc. of 36th IEEE-CDC*, Kobe (Japan), pp. 992–993.

Castelan, E. B. and Tarbouriech, S. (2000). Weak and strong Δ-invariant polyhedral sets for discrete-time singular systems, *Anais do XIII Congresso Brasileiro de Automática (CBA 2000)*, Florianópolis (Brazil).

Dai, L. (1989). *Singular Control System*, Springer-Verlag.

De Santis, E. (1994). On positively invariant sets for discrete-time linear systems with disturbance: an application of maximal disturbance sets, *IEEE Transactions on Automatic Control* **39**(1): 245–249.

Georgiou, G. and Krikelis, N. J. (1991). A design approach for constrained regulation in discrete singular systems, *System & Control Letters* **17**: 297–304.

Gilbert, E. G. and Tan, K. T. (1991). Linear systems with state and control constraints : the theory and application of maximal output admissible sets, *IEEE Transactions on Automatic Control* **36**(9): 1008–1020.

Hennet, J. C. (1989). Une extension du lemme de farkas et son application au problème de régulation linéaire sous contraintes, *C. R. Acad. Sci. Paris* **308**(I): 415–419.

Hennet, J. C. and Béziat, J. P. (1991). A class of invariant regulators for the discrete-time linear constrained regulation problem, *Automatica* **27**(3): 549–554.

Kailath, T. (1980). *Linear Systems*, Prentice-Hall, Englewood Cliffs, NJ.

Kolmanovski, I. and Gilbert, E. G. (1995). Maximal output admissible sets for discrete-time systems with disturbance inputs, *Proc. of ACC 1995*, Seattle (USA), pp. 1995–1999.

Lewis, F. L. (1986). A survey of linear singular systems, *Circuits Systems Signal Process* **5**(1): 3–36.

Milani, B. E. A. and Dórea, C. E. T. (1996). On invariant polyhedra of continuous-time systems subject to additive disturbances, *Automatica* **35**(5): 785–789.

Tarbouriech, S. and Castelan, E. B. (1993). On positively invariant sets for singular discrete-time systems, *International Journal Systems Science* **24**(9): 1687–1705.

Tarbouriech, S. and Castelan, E. B. (1995). An eigenstructure assignment approach for constrained linear continuous-time systems, *System & Control Letters* **24**: 333–343.

Tarbouriech, S. and Castelan, E. B. (1997). Maximal admissible polyhedral sets for discrete-time singular systems with additive disturbances, *Proc. of 37th IEEE-CDC*, San Diego (USA), pp. V.4, 3164–3169.

Tarbouriech, S. and Gomes da Silva Jr., J. M. (1997). Admissible polyhedra for discrete-time system with saturating controls, *Proc. of ACC 1997*, Albuquerque (USA), pp. 3915–3919.