On Some Classical and Quantum Effects Due to Gravitational Fields

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We consider the gravitational fields generated by a cosmic string, a global monopole and a tubular matter with interior magnetic field (Safko-Witten space-time), and examine some classical and quantum effects due to these fields. We investigate the Aharonov-Bohm effect in the space-time of a cosmic string, using the loop variables. In the space-time of a global monopole, we calculate the total energy radiated by a uniformly moving charged scalar particle, for small solid angle deficit. We show that the radiated energy is proportional to the cube of the velocity of the particle and to the cube of the Lorenz factor, in the non-relativistic and ultra-relativistic cases, respectively. In the Safko-Witten space-time, we investigate the existence of an electrostatic self-force on a charged particle. We also consider a hydrogen atom in the background space-time generated by a cosmic string and we find the solutions of the corresponding Dirac equation and we determine the energy levels of the atom. We investigate how the topological features of this space-time lead to shifts in the energy levels as compared with the flat Minkowski space-time. We study the behavior of non-relativistic quantum particles interacting with a Kratzer potential in the space-time generated by a global monopole and we find the energy spectrum in the presence of this topological defect. In the Safko-Witten space-time, an investigation is also made concerning the interaction of an harmonic oscillator with this background gravitational field.

Keywords: Cosmic string; Gravitational fields; Cosmology

I. INTRODUCTION

The general theory of relativity, as a metric theory, predicts that gravitation is manifested as curvature of space-time, which is characterized by the Riemann tensor. On the other hand, we know that there are connections between topological properties of the space and local physical laws in such a way that the local intrinsic geometry of the space is not sufficient to describe completely the physics of a given system. As an example of a gravitational effect of topological origin, we can mention the fact that only when a particle is transported around a cosmic string[1]-[3] along a closed curve the string is noticed at all. This situation corresponds to the gravitational analogue[4] of the electromagnetic Aharonov-Bohm effect[5], in which electrons are beamed past a solenoid containing a magnetic field. These effects are of topological origin rather than local. In fact, the nontrivial topology of space-time, as well as its curvature, leads to a number of interesting gravitational effects. Thus, it is also important to investigate the role played by a nontrivial topology. As examples of these investigations we can mention the study of some classical effects produced by the nontrivial topology of the cosmic string space-time as the gravitational lensing[3], emission of radiation by a freely moving particle[6], electrostatic self-force[7, 8] on an electric charge at rest and the so-called gravitational Aharonov-Bohm effect[4] among other. As to the quantum effects, we can mention the topological scattering [9], the interaction of a quantum system with conical singularities[10, 11] and with topological defects space-times[12]. As a result of these investigations, we conclude that it is important to take into account the topology of the background space-times in order to describe completely and precisely the physical contents of a given system.

Therefore, the problem of finding how a physical system placed in a gravitational field is influenced by the background space-time has to take into account the geometrical and topological features of the space-time under consideration and in this way we should emphasize that when a physical system is embedded in a curved space-time it is influenced by its geometry and topology.

According to standard quantum mechanics, the motion of a charged particle can be influenced by electromagnetic fields in regions from which the particle is rigorously excluded[5]. In this region the electromagnetic field vanishes. This phenomenon has come to be called Aharonov-Bohm effect[5]. The analogue of the electromagnetic Aharonov-Bohm effect set up is the background space-time of a cosmic string[1]-[3] in which the geometry is flat everywhere apart from a symmetry axis. In the space-time of a cosmic string we will investigate this phenomenon using the loop variables.

In the framework of Quantum Electrodynamics, the Bremsstrahlung process corresponds to the emission of radiation by a charged particle when it changes its momentum in collision with obstacles such as other particles or when it is accelerated due to the presence of electromagnetic fields. Therefore, in flat space-time particles moving freely do not radiate. On the other hand, in curved space-times the situation is quite different, and in this case, a charged particle moving on geodesic does radiate. This corresponds to the Bremsstrahlung process produced by gravitational fields and this may arises due to the curvature, topology or due to the combined effects of the geometric and topological features of the space-time.

We will consider the problem concerning the emission of radiation by a freely moving particle[13], caused by the combined effects of the geometrical and topological features of the space-time generated by a point-like global monopole[14]. The origin of this radiation is associated with the geometrical and topological features of the global monopole space-time which produces an effect proportional to the solid angle deficit.

It is well known that a charged particle placed in a curved space-time, even at the rest, experiences a self-force due to the geometrical and topological features of the space-time. In particular, for a conical space-time, it is entirely due to the nonlocal structure of the gravitational field[2, 8]. In this paper we will calculate the self-force on a charged particle at rest in the space-time of Safko-Witten[15], and show that its origin is exclusively due to the nontrivial topology of this space-time.

The study of quantum systems in curved space-times goes back to the end of twenties and to the beginning of thirties of the last century[16], when the generalization of the Schrödinger and Dirac equations to curved spaces has been discussed, motivated by the idea of constructing a theory which combines quantum physics and general relativity.

Spinor fields and particles interacting with gravitational fields has been the subject of many investigations. Along this line of research we can mention those concerning the determination of the renormalized vacuum expectation value of the energy-momentum tensor and the problem of creation of particles in expanding Universes[17], and those connected with quantum mechanics in different background space-times[18] and, in particular, the ones which consider the hydrogen atom in an arbitrary curved space-time[19, 20].

It has been known that the energy levels of an atom placed in a gravitational field will be shifted as a result of the interaction of the atom with space-time curvature[19, 20]. These shifts in the energy levels, which would depend on the features of the space-time, are different for each energy level, and thus are distinguishable from the Doppler effect and from the gravitational and cosmological red-shifts, in which cases these shifts would be the same for all spectral lines. In fact, it was already shown that in the Schwarzschild geometry, the shift in the energy level due to gravitational effects is different from the Stark and Zeeman effects, and therefore, it would be possible, in principle, to separate the shifts in the energy levels caused by electromagnetic and by gravitational perturbations[20]. Thus, in these situations the energy spectrum carries unambiguous information about the local features of the background space-time in which the atomic system is located.

In this paper in which concerns quantum effects due to gravitational fields, we deal with the interesting problem related with the modifications of the energy levels of a quantum system placed in the gravitational fields of a cosmic string, a global monopole and a tubular matter source with interior magnetic field(Safko-Witten space-time). In order to investigate this problem further, firstly, we determine the solutions of the corresponding Dirac equations and the energy levels of a hydrogen atom under the influence of the gravitational field of a cosmic string. To do these calculations we shall make the following assumptions: (i) The atomic nucleus is not affected by the presence of the defect. (ii) The atomic nucleus is located on the defect. With these, to do our calculations accordingly would have been possible and doing so it affords an explicit demonstration of the effects of space-time topology on the shifts in the atomic spectral lines of the hydrogen atom.

An atom placed in a gravitational field will be influenced by its interaction with the local curvature as well as with the topology of the space-time. As a result of this interaction, an observer at rest with respect to the atom would see a change in its spectrum. This shift in the energy of each atomic level would depend on the features of the space-time. The problem of finding these shifts[20] in the energy levels under the influence of a gravitational field is of considerable theoretical interest as well as possible observational.

In the global monopole space-time, we will investigate the interactions of a non-relativistic quantum particle with the Kratzer potential in this background space-time. In this case we also determine the shifts in the energy levels. Other investigations concerning quantum systems in this background space-time were done recently [21,22]. We will also investigate the quantum effects of gravitational fields in the framework of the Safko-Witten space-time in which case we consider the harmonic oscillator and determine how the nontrivial topology of this background space-time perturbs the energy spectrum. In this case, the influence of the conical geometry on the energy eigenvalues manifests as a kind of gravitational Aharonov-Bohm effect[4].

This paper is organized as follows. In Section II we present some classical effects, namely, the gravitational Aharonov-Bohm effect due to a cosmic string, the emission of radiation by a freely moving particle in the space-time of a global monopole, and the existence of a finite electrostatic self-force on a charged particle, at rest, in the space-time of Safko-Witten. In Section III, we present some quantum effects, as for example, the energy shifts in the hydrogen atom placed in the gravitational field of a cosmic string, the modifications of the spectra of a particle in the presence of the Kratzer potential in the space-time of a global monopole, and the modifications of energy spectrum of an harmonic oscillator in the space-time of Safko-Witten. Finally, in Section IV, we end up with some final remarks.

Throughout this paper we will use units in which $c = \hbar = 1$

A. Gravitational field of a cosmic string

The space-time produced by a static straight cosmic string can be obtained in the weak-field limit(valid for $G\overline{\mu} << 1$, where $\overline{\mu}$ is the linear mass density of the string). In this approximation, we write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{1}$$

where $\eta_{\mu\nu} = diag(1, -1, -1 - 1)$ is the Minkowski metric and $|h_{\mu\nu}| << 1$. Then, the solution of Einstein equations can be put into the form[1]

$$ds^2 = dt^2 - d\rho^2 - (1 - 4G\overline{\mu})^2 \rho^2 d\theta^2 - dz^2, \tag{2}$$

in a cylindrical coordinate system (t, ρ, φ, z), with $\rho \geq 0$ and $0 \leq \varphi \leq 2\pi$.The parameter $\alpha = 1 - 4G\overline{\mu}$ runs in the interval $(0, 1]$, with $\overline{\mu}$ being the linear mass density of the cosmic string. This is a Minkowski space-time with a wedge removed. The metric can be transformed to a locally Minkowski form with $\theta' = (1-4G\bar{\mu})\theta$, but now $0 \leq \theta\prime \leq 2\pi(1-4G\bar{\mu})$. In the coordinates $(t,\rho,\theta\prime,z)$ geodesics are straight lines, but in coordinates (t,ρ,θ,z) they bend through an angle $4\pi G\bar{\mu}$.

The gravitational field of a cosmic string is quite remarkable; a particle placed at rest around a straight, infinite, static string will not be attracted to it; there is no local gravity. The space-time around a cosmic string is locally flat but not globally. The external gravitational field due to a cosmic string may be approximately described by a commonly called conical geometry. Due to this conical geometry a cosmic string can induce several effects like, for example, gravitational leasing [3], electrostatic self-force[7, 8] on an electric charge at rest, Bremsstrahlung process[6] and the so-called gravitational Aharonov-Bohm effect[4].

B. Gravitational field of a global monopole

A global monopole is a heavy object formed in the phase transition of a system composed by a self-coupling iso-scalar triplet ϕ^a, whose original $O(3)$ symmetry is spontaneously broken to $U(1)$ [14].

Combining this matter field with the Einstein equations and considering the general form of the metric with spherical symmetry

$$ds^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{3}$$

the gravitational field is solved and gives the following result[14]

$$B = A^{-1} = 1 - 8\pi\eta^2 - 2\frac{M}{r}, \tag{4}$$

where $M \sim M_{core}$ and η is the symmetry-breaking scale. It is worth noticing that far away from the global monopole core the main effects are produced by the solid angle deficit and thus we can neglect the monopole's mass. Therefore, we obtain the metric of a point-like global monopole which can be written as[14]

$$ds^2 = b^2dt^2 - b^{-2}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{5}$$

where the parameter b is connected with the energy scale of symmetry breaking η and is given by the relation $b^2 = 1-8\pi\eta^2$. For typical grand unified theory the parameter η is of order $10^{16}GeV$ and thus $1-b^2 = 8\pi\eta^2 \sim 10^{-5}$. The space-time (5) is the solution of Einstein equations with diagonal energy momentum tensor with components $T^{\mu}_{\nu} = diag(2,2,1,1)(b^2-1)/r^2$.

Rescaling the time and radial coordinates by relations $t \to t/b^2$ and $r \to r/b^2$ we obtain the following form for the line element

$$ds^2 = dt^2 - dr^2 - b^2r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{6}$$

which will be used in what follows. Here $r \in [0,\infty]$, $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$.

This metric corresponds to a space-time with a solid angle deficit $\Delta = 32\pi^2G\eta^2$; test particles are deflected (topological scattering) by an angle $\pi\frac{\Delta}{2}$ irrespective to their velocity and impact parameter. This metric represents a curved space-time whose curvature vanishes in the case $b = 1$ (flat space-time). For $\theta = \frac{\pi}{2}$, the metric (6) is exactly the same as that of a cosmic string, in which case the azimuthal angle φ has a deficit $\Delta = 2\pi(1-b)$.

C. Safko-Witten space-time

The gravitational field due to a tubular matter source with an axial interior magnetic field and vanishing exterior magnetic field[15], which we are calling (Safko-Witten space-time) is introduced in this subsection. This space-time corresponds to a solution of the combined Einstein-Maxwell field with cylindrical symmetry. The exterior space-time corresponding to this configuration of fields is locally flat, but globally it is not flat. The geometry of the section $t = constant$ and $z = constant$ is a cone. The line element corresponding to this case is[15]

$$ds^2 = e^{2\beta}\left(dt^2 - d\rho^2\right) - \rho^2 d\varphi^2 - dz^2 \tag{7}$$

The parameter β is given by

$$\beta = \ln\left(\frac{\left(1+H_i\rho_1^2\right)^2}{\rho^2}\right) + \frac{4H_i\rho_1}{1+H_i\rho_1^2}\frac{(\rho_2-\rho_1)}{(\gamma+1)}$$

and depends on the intensity of the interior magnetic field through H_i and on the mass. The quantities ρ_1 and ρ_2 are the interior and exterior radii of the tube of matter and γ is an arbitrary constant.

This space-time corresponds to a Minkowski space-time minus a wedge with deficit angle $2\pi e^{-\beta}$ as we can see by defining the coordinates t', ρ', φ' by $t' = e^{\beta}t$, $\rho' = e^{\beta}\rho$ and $\varphi' = e^{-\beta}\varphi$. In this space-time, any observer outside the tube of matter would see a flat space. This local flatness means that there is no local gravity due to the tubular matter source with interior magnetic field, however we have some very interesting gravitational effects associated with the non-trivial topology of the space-like section.

II. CLASSICAL EFFECTS

In this section, we will consider some gravitational effects at classical level, due to the gravitational fields generated by a cosmic string, a global monopole and a tubular matter source with an interior magnetic field. The effects under consideration will be the gravitational analogue of the electromagnetic Aharonov-Bohm effect, the emission of radiation by a freely moving particle and the electrostatic self-force on a charged particle, in the cosmic string, global monopole and Safko-Witten space-times, respectively.

A. Gravitational Aharonov-Bohm effect due to a cosmic string

In the sixties and seventies of last century, it was proposed[23, 24] a new formalism for gravitation in which the fields depend on paths rather than space-time points. In this formalism several equations were established involving these new variables(loop variables). The loop variables were also used to describe gravitation[25], and put into evidence the equivalence between Einstein equations and the corresponding ones obtained using the loop variables. These variables were also computed[26] for the gravitational field corresponding to the Kerr metric. Other investigations related to the use of loop variables in some gravitational fields were done recently[27].

The loop variables in the theory of gravity are matrices representing parallel transport along curves in a space-time with a given affine connection.They are connected with the holonomy transformation which contains important topological informations. These mathematical objects contain information, for example, about how vectors change when parallel transported around a closed curve.They are defined as the limit of an ordered product of matrices of infinitesimal parallel transport as

$$U^{\mu}_{\nu}(C_{yx};\Gamma) \equiv \prod_{i=1}^{N}(\delta^{\rho_i}_{\rho_{1-i}} - \Gamma^{\rho_i}_{\lambda_i\rho_{1-i}}(x_i))dx^{\lambda_i}_i, \tag{8}$$

where $x_0 = x$, $\rho_o = \nu$, $x_N = y$, $\rho_N = \mu$, $dx_i = (x_i - x_{i-1})/\varepsilon$.

The points x_i lie on an oriented curve C_{yx} with its beginning at the point x and its end at the point y. The parallel-transport matrix U^{μ}_{ν} is a functional of the curve C_{yx} as a geometrical object.

If we choose a tetrad frame, a basis $\left\{e^{(a)}_{\mu}(x)\right\}$ and a loop C such that $C(0) = C(1) = x$, then in parallel transporting a vector X^{α} from $C(\lambda)$ to $C(\lambda + d\lambda)$, the vector components change by

$$\delta X^{\mu} = M^{\mu}_{\nu}[x(\lambda)]\, X^{\nu} d\lambda, \tag{9}$$

where $M^{\ \mu}_{\nu}$ is an infinitesimal linear map which depends on the tetrads, on the affine connection of the space-time and on the value of λ.Then, it follows that the holonomy transformation U^{μ}_{ν} is given by the matrix product of the N linear maps

$$U^{\mu}_{\nu} = \lim_{N\to\infty} \prod_{i=1}^{\infty}\left[\delta^{\mu}_{\nu} + \frac{1}{N} M^{\mu}_{\nu}[x(\lambda)]\,|_{\lambda=\frac{i}{N}}\right]. \tag{10}$$

One often writes the linear map U^{μ}_{ν} given by Eq.(10) as

$$U(C) = P\exp\left(\int_C M\right), \tag{11}$$

where P means ordered product along a curve C. Equation(11) should be understood as an abbreviation of the right hand side of Eq.(10). Note that if M^{μ}_{ν} is independent of λ, then it follows from Eq.(10) that M^{μ}_{ν} is given by $M^{\mu}_{\nu} = (\exp M)^{\mu}_{\nu}$.

In this paper we shall use the following notation

$$U_{AB}(C) = P\exp\left(\int_B^A \Gamma_{\mu}(x(\lambda))\frac{dx^{\mu}}{d\lambda}d\lambda\right), \tag{12}$$

where Γ_{μ} is the tetradic connection and A and B are the initial and final points of the path. Then, associated with every path C from a point A to a point B, we have a loop variable U_{AB} given by Eq.(12) which, by construction, is a function of the path C as a geometrical object.

In what follows we will compute the loop variables in the theory of gravity on the basis of a metric formalism, for different curves in the cosmic string space-time. In order to do this, let us consider a coordinate system $x^0 = t$, $x^1 = \rho$, $x^2 = \varphi$, $x^3 = z$ and define the one-forms $\omega^a (a = 0,1,2,3)$ as

$$\begin{aligned}
\omega^0 &= dt, && (13)\\
\omega^1 &= \cos\varphi d\rho - \alpha\rho\sin\varphi d\varphi, && (14)\\
\omega^2 &= \sin\varphi d\varphi + \alpha\rho\cos\varphi d\varphi, &&\\
\omega^3 &= dz && (15)
\end{aligned}$$

Using the Cartan structure equations $d\omega^a = -\omega^a_b \wedge \omega^b = e^{(a)}_{\mu\|\nu}dx^{\nu}dx^{\mu}$, we get the following result for the tetradic connections

$$-\Gamma^1_{\mu 2}dx^{\mu} = \Gamma^2_{\mu 1}dx^{\mu} = -(1-\alpha)d\varphi \tag{16}$$

Let us consider a circle C in the equator $(d\rho = dt = d\varphi = 0)$, then the holonomy is given by

$$U(C) = \exp\left(\int_0^{2\pi}\Gamma_{\varphi}d\varphi\right) = e^{-8\pi i\bar{\mu}J_{12}}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \cos 8\pi\bar{\mu} & \sin 8\pi\bar{\mu} & 0\\ 0 & -\sin 8\pi\bar{\mu} & \cos 8\pi\bar{\mu} & 0\\ 0 & 0 & 0 & 1\end{pmatrix}, \tag{17}$$

where J_{12} is the generator of rotations about the z-axis.

¿From the above result we see that when we parallel transport a vector around the cosmic string at rest at the origin, this vector acquires a phase that comes from the holonomy which is given by $\exp(-8i\pi\bar{\mu}J_{12})$. If now we consider a segment in the z-direction, translation in time, and radial segments, we conclude that in all these cases the holonomies vanish identically. Therefore, we can use these results, and write a general expression for $U(C)$. In the general case, $U(C)$ reads[28]

$$U(C) = P\exp\left(-\frac{i}{2}\int_C \Gamma^{ab}_{\mu}J_{ab}dx^{\mu}\right), \tag{18}$$

where J_{ab} are the generators of the Lorentz group $SO(3,1)$ and Γ^{ab}_{μ} are the appropriate tetradic connections. Therefore, we can say that when we carry a vector along curves in this space-time, it acquires a phase that depends on α(or $\bar{\mu}$), which prevents it from being equal to the unit matrix. This effect is exclusively due to the non-trivial topology of the cosmic string space-time. This is a gravitational analogue[4] of the Aharonov-Bohm effect[5] but, in this case, purely at classical level.

B. Emission of radiation by a particle in the gravitational field of a global monopole

Now, let us consider a scalar particle with scalar charge q living in the space-time of a global monopole. The scalar and minimal coupling field corresponding to this particle obeys the Klein-Gordon equation

$$\Box\Phi(x) = -4\pi j(x), \tag{19}$$

with a scalar current

$$j(x) = q\int \delta^4(x - x(\tau))\frac{d\tau}{\sqrt{-g}} =$$

$$\frac{q}{u^0}\frac{\delta(r-r(t))\delta(\varphi-\varphi(t))\delta(\theta-\theta(t))}{b^2r^2\sin^2\theta}. \tag{20}$$

The trajectory of a freely moving particle in this space-time may be found in general form from the standard set of equations of geodesic line. For simplicity and due to spherical symmetry we consider the trajectory of the particle in the plane $\theta = \frac{\pi}{2}$, assuming that at time $t = 0$ the particle is in the closest distance ρ from monopole's core, which is by definition, the impact parameter. The trajectory has the following form

$$r = \sqrt{\rho^2 + v^2t^2},\ \varphi = \frac{1}{b}\arctan\frac{vt}{\rho},\ \theta = \frac{\pi}{2},\ u^0 \equiv \gamma = \frac{1}{\sqrt{1-v^2}}, \tag{21}$$

where v is the constant velocity of the particle and u^0 is the zero component of the four velocity.

To find the total energy radiated by the particle during all its history we adopt an approach already used[6]. Let us summarize here its main aspects. The total energy radiated by a particle is expressed in terms of the covariant divergence of the energy-momentum tensor as follows

$$\mathcal{E} = \int T^{\nu}_{\mu;\nu}\xi^{\mu}\sqrt{-g(x)}d^4x, \tag{22}$$

where ξ^{μ} is the time-like Killing vector. Taking into account the explicit form of the energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4\pi}\left(\Phi_{,\mu}\Phi_{,\nu} - \frac{1}{2}g_{\mu\nu}\Phi_{,\alpha}\Phi^{,\alpha}\right), \tag{23}$$

the equation of motion for a minimally coupled scalar field (19) and the explicit expression for the Killing vector $\xi^{\mu} = (1,0,0,0)$, one has the following expression for the total energy radiated by the particle during all time

$$\mathcal{E} = 4\pi\int \frac{\partial}{\partial t}D^{rad}(x;x')j(x)j(x')\sqrt{-g(x)}\sqrt{-g(x')}d^4xd^4x', \tag{24}$$

where

$$D^{rad}(x;x') = \frac{1}{2}\left[D^{ret}(x;x') - D^{adv}(x;x')\right] \tag{25}$$

is the radiative Green function.

In order to find the Green's functions, let us first of all obtain the complete set of eigenfunction of the Klein-Gordon equation (19) which can be written as

$$\Box\Phi = \left\{\partial_t^2 - \frac{1}{r^2}\partial_r(r^2\partial_r) + \frac{1}{b^2r^2}\hat{L}^2\right\}\Phi = \lambda^2\Phi, \tag{26}$$

with eigenvalues λ^2. Here $\hat{L}^2$ is the square of the angular momentum operator. The complete set of solution of the eq. (26) was considered in the context of quantum fields[29], and it has the following form

$$\Phi_{l,m,\omega,p}(t,r,\theta,\varphi) = e^{-i\omega t}\sqrt{\frac{p}{2\pi b^2 r}}J_{\nu_l}(pr)Y_l^m(\theta,\varphi), \tag{27}$$

where $J_\nu(x)$ is the Bessel function of first kind; $Y_l^m(\theta,\varphi)$ is the spherical function $(l = 0,1,2,\cdots,\ |m| \le l))$; $p = \sqrt{\lambda^2+\omega^2}$ and

$$\nu_l = \sqrt{\frac{l(l+1)}{b^2} - \frac{1}{4}}. \tag{28}$$

Using this set of solutions the radiative Green function reads

$$D^{rad}(x;x') = \frac{i}{2b^2}\frac{1}{\sqrt{rr'}}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}Y_l^m(\theta,\varphi)Y_l^{m*}(\theta',\varphi')$$

$$\int_{-\infty}^{+\infty}d\omega\ \mathrm{sgn}(\omega)e^{-i\omega(t-t')} \tag{29}$$
$$\times\int_0^{\infty}dp\,pJ_{\nu_l}(pr)J_{\nu_l}(pr')\delta(p^2-\omega^2).$$

Taking into account this formula into Eq.(24), we obtain the following expression for the total energy

$$\mathcal{E} = \frac{2\pi q^2}{\gamma^2b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left|Y_l^m(\frac{\pi}{2},0)\right|^2\int_{-\infty}^{+\infty}d\omega\ |\omega|$$

$$\int_0^{\infty}dp\,p\delta(p^2-\omega^2)\left|S_l^m(\omega,p,v,\rho)\right|^2, \tag{30}$$

where we have introduced the function S_l^m by the relation

$$S_l^m(\omega,p,v,\rho) = \int_{-\infty}^{+\infty}dt e^{i\omega t - i\frac{m}{b}\arctan\frac{vt}{\rho}}\frac{J_{\nu_l}(p\sqrt{\rho^2+v^2t^2})}{(\rho^2+v^2t^2)^{1/4}}. \tag{31}$$

This function obeys the following symmetry relation

$$S_l^m(-\omega,p,v,\rho) = S_l^{-m}(\omega,p,v,\rho). \tag{32}$$

Using this we may represent the total energy as an integral

$$\mathcal{E} = \int_0^{\infty}d\omega\frac{d\mathcal{E}}{d\omega}, \tag{33}$$

where the spectral density is

$$\frac{d\mathcal{E}}{d\omega} = \omega\frac{2\pi q^2}{\gamma^2 b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left|Y_l^m(\frac{\pi}{2},0)\right|^2 |S_l^m(\omega,\omega,v,\rho)|^2 . \quad (34)$$

The function $S_l^m(\omega,\omega,v,\rho)$ given by Eq.(31) may be represented in a slightly different form, more suitable for analysis (here we assume $\omega > 0$) as

$$S_l^m(\omega,\omega,v,\rho) = -2\frac{\sqrt{\rho}}{v}\sin\frac{\pi}{2}\left[\nu_l - \frac{m}{b} - \frac{1}{2}\right]\tilde{S}_l^m(\omega,v,\rho), \quad (35)$$

where

$$\tilde{S}_l^m(\omega,v,\rho) = \int_1^{\infty} dy e^{-\frac{\omega\rho}{v}y}\left(\frac{y-1}{y+1}\right)^{-\frac{m}{2b}}\frac{I_{\nu_l}(\omega\rho\sqrt{y^2-1})}{(y^2-1)^{1/4}}. \quad (36)$$

Therefore we can express the spectral density of radiation by

$$\frac{d\mathcal{E}}{d\omega} = \omega\rho\frac{8\pi q^2}{v^2\gamma^2 b^2}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left|Y_l^m(\frac{\pi}{2},0)\right|^2\left|\tilde{S}_l^m(\omega,v,\rho)\right|^2\sin^2\frac{\pi}{2}\left[\nu_l - \frac{m}{b} - \frac{1}{2}\right]. \quad (37)$$

Integrating over the frequency ω, using formula 6.612(3) from Ref.[30], we find that the total energy is

$$\begin{aligned}\mathcal{E} &= -\frac{8q^2}{v^3\gamma^2 b^2\rho}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\left|Y_l^m(\frac{\pi}{2},0)\right|^2\sin^2\frac{\pi}{2}\left[\nu_l - \frac{m}{b} - \frac{1}{2}\right] \quad (38)\\ &\times\int_1^{\infty}\frac{dy}{y^2-1}\left(\frac{y-1}{y+1}\right)^{-\frac{m}{2b}}\int_1^{\infty}\frac{dy'}{y'^2-1}\left(\frac{y'-1}{y'+1}\right)^{-\frac{m}{2b}}(y+y')Q'_{\nu_l-\frac{1}{2}}[\cosh\sigma].\end{aligned}$$

Here $Q_\nu[x]$ is the Legendre function of second kind; the prime means the derivative with respect to its argument, and

$$\cosh\sigma = \frac{(y+y')^2 v^{-2} - y^2 - y'^2 + 2}{2\sqrt{y^2-1}\sqrt{y'^2-1}}.$$

Now, let us analyze the above expressions in the Minkowski limit. In this case we have to put $b = 1$. Due to this, the argument of sine is $\frac{\pi}{2}(l-m)$. Next we have to take into account that $Y_l^m(\frac{\pi}{2},0) = 0$, if $(l+m)$ is odd. For this reason the argument of sine is $\frac{\pi}{2}\times$ (even number) which implies that the sine of this quantity is zero and as a consequence the total energy is zero, too, as it must be in Minkowski space-time. Differently from the cosmic string space-time there is no specific values of b for which total energy is identically zero.

Let us simplify our formulas for the global monopole space-time assuming that solid angle deficit is small. In this case we can expand sine in the previous formulas in terms of b as

$$\sin^2\frac{\pi}{2}\left[\nu_l - \frac{m}{b} - \frac{1}{2}\right] \approx (1-b)^2\frac{\pi^2}{4}\left[\frac{l(l+1)}{l+\frac{1}{2}} - m\right]^2 . \quad (39)$$

Therefore up to $(1-b)^2$ we may set $b = 1$ in the rest part. Firstly, let us analyze the total energy given by Eq.(38). The sum over m can be made using the addition theorem for Legendre function of the first kind from[30] and results in

$$\begin{aligned}\mathcal{E} &= -\frac{\pi q^2(1-b)^2}{v^3\gamma^2\rho}\int_1^{\infty}\frac{dy}{y^2-1}\int_1^{\infty}\frac{dy'}{y'^2-1}(y+y')\sum_{l=0}^{\infty}\left[\left(l+\frac{1}{2}\right)^3 - \frac{1}{2}\left(l+\frac{1}{2}\right) + \frac{1}{16}\frac{1}{l+\frac{1}{2}}\right. \quad (40)\\ &+ \left.2\left(\left(l+\frac{1}{2}\right)^2 - \frac{1}{4}\right)\partial_\beta + \left(l+\frac{1}{2}\right)\partial_\beta^2\right]P_l[\cosh\beta]Q'_l[\cosh\sigma],\end{aligned}$$

where

$$\cosh\beta = \frac{yy'+1}{\sqrt{y^2-1}\sqrt{y'^2-1}}. \quad (41)$$

Using now an integral representation for the Legendre function of the second kind as below

$$Q_l[\cosh\sigma] = \frac{1}{\sqrt{2}}\int_\sigma^\infty \frac{e^{-(l+1/2)t}dt}{\sqrt{\cosh t - \cosh\sigma}}, \qquad (42)$$

and the relation

$$\sum_{l=0}^\infty e^{-(l+1/2)t}P_l[\cosh\beta] = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{\cosh t - \cosh\beta}}, \qquad (43)$$

we get, finally, the following expression for the total energy

$$\mathcal{E} = -(1-b)^2\frac{\pi q^2 v\gamma^2}{2\rho}\int_1^\infty\int_1^\infty \frac{dydy'}{(y+y')^3}E(s,y,y'), \qquad (44)$$

where

$$\begin{aligned} E &= 1-48s^2z_1-192s^4z_2^2-16s^2z_2\int_0^\infty \frac{dx}{\sqrt{x}}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{R_x}}\left\{\frac{1}{(x+1)^{3/2}}+\frac{6s^2z_1}{(x+1)^{5/2}}+\frac{15}{2}\frac{s^4z_2^2}{(x+1)^{7/2}}\right\}\right] \\ &+ \frac{1}{4}\int_0^\infty \frac{dx}{\sqrt{x}}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{R_x}}\int_x^\infty \frac{dx'}{\sqrt{R_{x'}}}\frac{1}{\sqrt{x'+1}}\right], \end{aligned} \qquad (45)$$

and we have introduced the following definitions

$$\begin{aligned} z_1 &= \frac{yy'+1}{(y+y')^2},\ z_2 = \frac{1}{y+y'}, \\ R_x &= (x+1)^2+4s^2z_1(x+1)+4s^4z_2^2, \end{aligned}$$

with the parameter s given by $s = v\gamma$.

The formula obtained looks very awesome but it may be analyzed without great problem for non-relativistic and ultra-relativistic particles. The function E depends only on the combination $s = v\gamma = v/\sqrt{1-v^2}$. In the non- relativistic case the parameter $s\to 0$ and in the ultra-relativistic case $s\to\infty$, and therefore we have to analyze the function E in these two limits.

In the non-relativistic case we may expand all integrand in Eq.(45) over small values of the parameter s and calculate the integrals. The main contribution, in this case, is proportional to the cube of the velocity

$$\mathcal{E} = (1-b)^2\frac{\pi q^2}{2\rho}v^3. \qquad (46)$$

The ultra-relativistic case is more complicate due to the last term in Eq.(45). There is no need, in fact, to calculate the contribution from it. It is enough to find an upper bound for it. Let us analyze the contribution from this term which is given by

$$W = -(1-b)^2\frac{\pi q^2 v\gamma^2}{8\rho}\int_1^\infty\int_1^\infty\frac{dydy'}{(y+y')^3}\int_0^\infty\frac{dx}{\sqrt{x}}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{R_x}}\int_x^\infty\frac{dx'}{\sqrt{R_{x'}}}\frac{1}{\sqrt{x'+1}}\right]. \qquad (47)$$

First of all one represents the polynomial R in the following form

$$R_x = (x+1+s^2\delta_+^2)(x+1+s^2\delta_-^2), \qquad (48)$$

where $\delta_\pm^2 = 2(z_1\pm\sqrt{z_1^2-z_2^2})$. Because $\delta_\pm^2$ are positive we can write out the following inequalities

$$\int_x^\infty\frac{dx'}{\sqrt{R_{x'}}}\frac{1}{\sqrt{x'+1}} \le \int_0^\infty\frac{dx'}{\sqrt{R_{x'}}}\frac{1}{\sqrt{x'+1}} \le \int_0^\infty\frac{dx'}{(x+1)^{3/2}} \le 2. \qquad (49)$$

Using this upper bound we have

$$|W| \le (1-b)^2\frac{\pi q^2 v\gamma^2}{4\rho}\int_1^\infty\int_1^\infty\frac{dydy'}{(y+y')^3}\frac{\mathbf{E}(\sqrt{1-\frac{b^2}{a^2}})}{ab^2} \le (1-b)^2\frac{\pi^2q^2v\gamma^2}{8\rho}\int_1^\infty\int_1^\infty\frac{dydy'}{(y+y')^3}\frac{1}{ab^2}, \qquad (50)$$

where $a = \sqrt{1+s^2\delta_+^2}$, $b = \sqrt{1+s^2\delta_-^2}$ and we used the fact that the upper bound for elliptic integral of second kind $\mathbf{E}$ is

$\pi/2$. Now we change the variables $y \to sy$, $y' \to sy'$ and take the ultra-relativistic limit $s \to \infty$. In the end we have the following estimation

$$|W| \leq (1-b)^2 \frac{\pi^3 q^2}{32\rho}. \tag{51}$$

The contribution of others terms in Eq. (45) is of order larger than γ^3. Calculating the other integrals in Eq. (45) one has that the total energy radiated in ultra-relativistic case is given by

$$\mathcal{E} = (1-b)^2 \frac{3\pi^3 q^2}{32\rho} \gamma^3. \tag{52}$$

C. Force on a charged particle at rest in the Safko-Witten space-time

In this section we are interested in determining the electrostatic self-force on a point test charge q located at the point $\rho = \rho_0$, $\varphi = \pi$ and $z = 0$ in the space-time under consideration, which is locally flat, but with a non-trivial conical topology. As we will show, even with the curvature being zero in all points, except in the localization of the tube of matter and inside it, there is the interesting physical effect which corresponds to the existence of a self-force.

In order to obtain an expression for the self-force, we will determine the electrostatic potential $V(\rho,\varphi,z)$ generated by the point test charge. To do this let us use the Maxwell equations in curved space-time which are given by

$$\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\mu} \left(\sqrt{-g} F^{\mu\nu}\right) = -\frac{1}{\varepsilon_0} j^\nu \tag{53}$$

where

$$j^0 = \rho = \sum_n \frac{q_n}{\sqrt{-g^{(3)}}} \delta(\mathbf{r} - \mathbf{r}_n) \quad \text{and} \quad j^i = \sum_n \frac{q_n}{\sqrt{-g^{(3)}}} \delta(\mathbf{r} - \mathbf{r}_n) \frac{dx^i}{dz^0}$$

$g^{(3)}$ being the determinant of the spatial metric tensor.

¿From Eq.(53) we obtain the equation for the electrostatic potential due to a point test charge q. In the background metric given by Eq.(7) we get the following potential equation

$$\left(\frac{e^{-2\beta}}{\rho} \partial_\rho \left(\rho \partial_\rho\right) + \frac{1}{\rho^2} \partial_\varphi^2 + \partial_z^2\right) V(\rho,\varphi,z) = -\left(\frac{q}{\varepsilon_0}\right) \frac{\delta(\rho-\rho_0)\,\delta(\varphi-\pi)\,\delta(z)}{\rho e^\beta} \tag{54}$$

where ε_0 is the permittivity of free space.

Changing the variable $\rho \to e^{-\beta}\rho'$, we get

$$\left(\frac{e^{-2\beta}}{\rho'} \partial_{\rho'} \left(\rho' \partial_{\rho'}\right) + \frac{1}{\rho'^2} \partial_\varphi^2 + \partial_z^2\right) V(\rho',\varphi,z) = -\left(\frac{q}{\varepsilon_0}\right) \frac{\delta(\rho'-\rho'_0)\,\delta(\varphi-\pi)\,\delta(z)}{\rho' e^\beta} \tag{55}$$

where we used the property $\delta\left[e^{-\beta}(\rho'-\rho'_0)\right] = e^\beta \delta(\rho'-\rho'_0)$.

Equation (55) reduces to the usual potential equation, in the subset of Minkowski space-time covered by the coordinate system (t',ρ',θ,z), where $\theta = e^{-\beta}\varphi$, with the point charge located at $\rho' = \rho'_0$, $\theta = \pi e^{-\beta}$ and $z = 0$. In this case it can be written as

$$\left(\frac{1}{\rho'} \partial_{\rho'} \left(\rho' \partial_{\rho'}\right) + \frac{1}{\rho'^2} \partial_\theta^2 + \partial_z^2\right) V(\rho',\theta,z) = -\left(\frac{q}{\varepsilon_0}\right) \frac{\delta(\rho'-\rho'_0)\,\delta(\theta-\pi)\,\delta(z)}{\rho'}. \tag{56}$$

The potential $V(\rho',\theta,z)$ must satisfy the boundary conditions

$$\begin{aligned} V(\rho',0,z) &= V(\rho',2\pi e^{-\beta},z) \\ \frac{\partial V}{\partial\theta}(\rho',0,z) &= \frac{\partial V}{\partial\theta}(\rho',2\pi e^{-\beta},z). \end{aligned} \tag{57}$$

Following the same procedure adopted by Linet[7] who used a result[31] on the electrostatics for a wedge formed from two semi-infinite conducting planes, we can write the solution of Eq.(56) as

$$V(\rho,\theta,z) = \frac{q}{(4\pi\varepsilon_0)\,2\pi e^{-\beta}\left(2\rho'\rho_0'\right)^{1/2}}\int_\eta^\infty\left(\frac{\sinh\left(\xi e^{\beta}/2\right)}{\cosh\left(\xi e^{\beta}/2\right)+\sin\left(\theta e^{\beta}/2\right)} + \frac{\sinh\left(\xi e^{\beta}/2\right)}{\cosh\left(\xi e^{\beta}/2\right)-\sin\left(\theta e^{\beta}/2\right)}\right)\frac{d\xi}{(\cosh\xi-\cosh\eta)^{1/2}} \tag{58}$$

where η is defined by $\cosh\eta = \left(\rho'^2+\rho_0'^2+z^2\right)/2\rho'\rho_0'$, $(\eta \geq 0)$.

Returning to coordinate φ, which is related with θ by $\theta = e^{-\beta}\varphi$, we obtain the following expression for the electrostatic potential

$$V(\rho',\varphi,z) = \frac{q}{(4\pi\varepsilon_0)\,2\pi e^{-\beta}\left(2\rho'\rho_0'\right)^{1/2}}\int_\eta^\infty \frac{\sinh\left(\xi e^{\beta}\right)d\xi}{\left[\cosh\left(\xi e^{\beta}\right)+\cos\varphi\right](\cosh\xi-\cosh\eta)^{1/2}} \tag{59}$$

In the neighborhood of the point charge, we can separate the electrostatic potential (59) in two terms as follows

$$V(\rho',\varphi,z) = V_M(\rho',\varphi,z) + H(\rho',\varphi,z) \tag{60}$$

where the first term, V_M, is infinite at the position of the charge and corresponds to the Coulomb potential in Minkowski space-time, and H is a regular solution of the homogeneous equation corresponding to Eq.(55).

Then, we have

$$V_M(\rho',\varphi,z) = \frac{q}{4\pi\varepsilon_0\left\{\rho'^2+\rho_0'^2+z^2-2\rho'\rho_0'\left[\cos\left(e^{-\beta}(\varphi-\pi)\right)\right]\right\}^{1/2}}. \tag{61}$$

Equation (61) reduces to the electrostatic potential due to a point charge in the flat space located at $\rho' = \rho_0'$, $\varphi = \pi$ and $z = 0$ when β is equal to zero.

In order to obtain $H(\rho',\varphi,z)$ it is necessary to write Eq.(61) in integral form as

$$V_M(\rho',\varphi,z) = \frac{q}{(4\pi\varepsilon_0)\,\pi\left(2\rho'\rho_0'\right)^{1/2}}\int_\eta^\infty \frac{\sinh\xi d\xi}{\left\{\cosh\xi-\cos\left[e^{-\beta}(\varphi-\pi)\right]\right\}(\cos\xi-\cos\eta)^{1/2}}. \tag{62}$$

Comparing Eq.(59) and (62), and using Eq.(61), we obtain the following expression for $H(\rho',\varphi,z)$

$$H(\rho',\varphi,z) = \frac{q}{(4\pi\varepsilon_0)\,\pi\left(2\rho'\rho_0'\right)^{1/2}} \times\int_\eta^\infty\left(\frac{\sinh\left(e^{\beta}\xi\right)}{e^{\beta}\left[\cosh\left(e^{\beta}\xi\right)+\cos\varphi\right]} - \frac{\sinh\xi}{\left(\cosh\xi-\cos\left[e^{\beta}(\varphi-\pi)\right]\right)}\right) \tag{63}$$

$$\times\frac{d\xi}{(\cos\xi-\cos\eta)^{1/2}}. \tag{64}$$

As locally we are in the Minkowski space-time, we will ignore the infinite forces arising from the electrostatic potential V_M, and we will consider $H(\rho',\varphi,z)$ as a kind of *external* electrostatic potential which exerts a force on the charge q. Then, we can compute the electrostatic force from the electrostatic energy which is given by

$$W = \frac{1}{2}qH\left(\rho_0',\pi,0\right) \tag{65}$$

at the point $\rho' = \rho_0'$, $\varphi = \pi$ and $z = 0$, at which the point charge is located.

From Eqs. (64) and (65), we obtain

$$W = \left(\frac{L_B}{4\pi}\right)\frac{q^2}{4\pi\varepsilon_0\rho_0'} \tag{66}$$

where L_B depends on the parameter β and is given by

$$L_B = \int_0^\infty \left(\frac{\sinh\left(e^{\beta}\xi\right)}{e^{-\beta}\left[\cosh\left(e^{\beta}\xi\right)-1\right]} - \frac{\sinh\xi}{\cosh\xi - -1} \right) \frac{d\xi}{\sinh\left(\frac{\xi}{2}\right)}. \tag{67}$$

Hence, from Eq.(66) the exerted force whose components are $f^{\rho} = -\partial W/\partial\rho_0'$, $f^{\varphi} = -\partial W/\partial\varphi$ and $f^z = -\partial W/\partial z$, is

$$f^{\rho} = \left(\frac{L_B}{4\pi}\right) e^{-2\beta} \frac{q^2}{4\pi\varepsilon_0\rho_0^2} \qquad f^{\varphi} = f^z = 0. \tag{68}$$

In the limit $\beta \to 0$, we get the following result

$$f^{\rho} \sim 0.2\beta \frac{q^2}{4\pi\varepsilon_0\rho_0^2}.$$

Therefore, in the space-time of a tubular matter source with interior axial magnetic field there is a self-interaction between a charged particle and this tube of matter induced by the non-trivial conical topology of this space-time.

III. QUANTUM EFFECTS

A. Relativistic hydrogen atom in the space-time of the cosmic string

In what follows we will consider the background space-time generated by a cosmic string and study the behavior of a hydrogen atom placed in it[32]. The line element corresponding to the cosmic string space-time is given, in spherical coordinates, by

$$ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - \alpha^2 r^2 \sin^2\theta d\phi^2. \tag{69}$$

Let us consider the generally covariant form of the Dirac equation which is given by

$$\left[i\gamma^{\mu}(x)\left(\partial_{\mu} + \Gamma_{\mu}(x) + ieA_{\mu}\right) - \mu\right]\Psi(x) = 0, \tag{70}$$

where μ is the mass of the particle, A_{μ} is an external electromagnetic potential and $\Gamma_{\mu}(x)$ are the spinor affine connections which can be expressed in terms of the set of tetrad fields $e^{\mu}_{(a)}(x)$ and the standard flat space-time $\gamma^{(a)}$ Dirac matrices as

$$\Gamma_{\mu} = \frac{1}{4}\gamma^{(a)}\gamma^{(b)} e^{\nu}_{(a)}\left(\partial_{\mu} e_{(b)\nu} - \Gamma^{\sigma}_{\mu\nu} e_{(b)\sigma}\right). \tag{71}$$

The generalized Dirac matrices $\gamma^{\mu}(x)$ satisfies the anti-commutation relations

$$\left\{\gamma^{\mu}(x), \gamma^{\nu}(x)\right\} = 2g^{\mu\nu}(x),$$

and are defined by

$$\gamma^{\mu}(x) = e^{\mu}_{(a)}(x)\gamma^{(a)}, \tag{72}$$

where $e^{\mu}_{(a)}(x)$ obeys the relation $\eta^{ab} e^{\mu}_{(a)}(x) e^{\nu}_{(b)}(x) = g^{\mu\nu}$; $\mu, \nu = 0,1,2,3$ are tensor indices and $a, b = 0,1,2,3$ are tetrad indices.

In this paper, the following explicit forms of the constant Dirac matrices will be taken

$$\gamma^{(0)} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix};\ \gamma^{(i)} = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix};\ i = 1,2,3, \tag{73}$$

where σ^i are the usual Pauli matrices.

In order to write the Dirac equation in this space-time, let us take the tetrads $e^{\mu}_{(a)}(x)$ as

$$e^{\mu}_{(a)}(x) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ 0 & \frac{\cos\theta\cos\phi}{r} & \frac{\cos\theta\sin\phi}{r} & -\frac{\sin\theta}{r} \\ 0 & -\frac{\sin\phi}{\alpha r\sin\theta} & \frac{\cos\phi}{\alpha r\sin\theta} & 0 \end{pmatrix}. \tag{74}$$

Thus using (74), we obtain the following expressions for the generalized Dirac matrices $\gamma^{\mu}(x)$

$$\begin{aligned} \gamma^0(x) &= \gamma^{(0)}, \\ \gamma^1(x) &= \gamma^{(r)}, \\ \gamma^2(x) &= \frac{\gamma^{(\theta)}}{r}, \\ \gamma^3(x) &= \frac{\gamma^{(\phi)}}{\alpha r\sin\theta}, \end{aligned} \tag{75}$$

where

$$\begin{pmatrix} \gamma^{(r)} \\ \gamma^{(\theta)} \\ \gamma^{(\phi)} \end{pmatrix} = \begin{pmatrix} \cos\phi\sin\theta & \sin\phi\sin\theta & \cos\phi \\ \cos\phi\cos\theta & \sin\phi\cos\theta & -\sin\phi \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \gamma^{(1)} \\ \gamma^{(2)} \\ \gamma^{(3)} \end{pmatrix}. \tag{76}$$

The covariant Dirac Eq. (70), written in the space-time of a cosmic string is then given by[32]

$$\left[i\Sigma^r\partial_r + i\frac{\Sigma^\theta}{r}\partial_\theta + i\frac{\Sigma^\phi}{\alpha r\sin\theta}\partial_\phi + i\frac{1}{2r}\left(1-\frac{1}{\alpha}\right)\left(\Sigma^r+\cot\theta\,\Sigma^\theta\right) - eA_0 - \gamma^{(0)}\mu + E\right]\chi(\vec{r}) = 0, \qquad (77)$$

where Σ^r, Σ^θ and Σ^ϕ are defined by

$$\Sigma^r \equiv \gamma^{(0)}\gamma^{(r)};\ \Sigma^\theta \equiv \gamma^{(0)}\gamma^{(\theta)};\ \Sigma^\phi \equiv \gamma^{(0)}\gamma^{(\phi)}, \qquad (78)$$

and we have chosen $\Psi(x)$ as

$$\Psi(x) = e^{-iEt}\chi(\vec{r}), \qquad (79)$$

which comes from the fact that the space-time under consideration is static.

We must now turn our attention to the solution of the equation for $\chi(\vec{r})$. Then, let us assume that the solutions of Eq. (77) are of the form

$$\chi(\vec{r}) = r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}(\sin\theta)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}R(r)\Theta(\theta)\Phi(\phi). \qquad (80)$$

Thus, substituting Eq.(80) into (77), we obtain the following radial equation

$$\left(c\Sigma'_r p_r + i\frac{\Sigma'_r}{r}\gamma^{(0)}k_{(\alpha)} + eA_0 + \mu\gamma^{(0)}\right)R(r) = ER(r). \qquad (81)$$

where

$$k_{(\alpha)} = \pm\left(j_{(\alpha)}+\frac{1}{2}\right) = \pm\left[j+\frac{1}{2}+m\left(\frac{1}{\alpha}-1\right)\right] \qquad (82)$$

are the eigenvalues of the generalized spin-orbit operator $K_{(\alpha)}$ in the space-time of a cosmic string and $j_{(\alpha)}$ corresponds to the eigenvalues of the generalized total angular momentum operator. The operator K_α is given by

$$\gamma^{(0)}K_{(\alpha)} = \vec{\Sigma}\cdot\vec{L}_{(\alpha)} + 1, \qquad (83)$$

with $\vec{\Sigma} = (\Sigma^r,\ \Sigma^\theta,\ \Sigma^\phi)$ $Y_1\alpha^m\alpha$ and $\vec{L}_{(\alpha)}$ is the generalized angular momentum operator[32] in the space-time of the cosmic string, which is such that $\vec{L}^2_{(\alpha)}Y^{m_{(\alpha)}}_{l_{(\alpha)}}(\theta,\phi) = l_{(\alpha)}\left(l_{(\alpha)}+1\right)Y^{m_{(\alpha)}}_{l_{(\alpha)}}$, with $Y^{m_{(\alpha)}}_{l_{(\alpha)}}(\theta,\phi)$ being the generalized spherical harmonics in the sense that $m_{(\alpha)}$ and $l_{(\alpha)}$ are not necessarily integers. The parameters $m_{(\alpha)}$ and $l_{(\alpha)}$ are given, respectively, by $m_{(\alpha)} \equiv \frac{m}{\alpha}$ and $l_{(\alpha)} \equiv n + m_{(\alpha)} = l + |m|\left(\frac{1}{\alpha}-1\right)$, $l = 0,1,2,\ldots\ n-1$, l is the orbital angular momentum quantum number, m is the magnetic quantum number and n is the principal quantum number.

Let us choose the following two-dimensional representation for Σ'_r and $\gamma^{(0)}$

$$\Sigma'_r \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \gamma^{(0)} \equiv \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \qquad (84)$$

Now, let us assume that the radial solution can be written as

$$R(r) = \frac{1}{r}\begin{pmatrix} -iF(r) \\ G(r) \end{pmatrix}. \qquad (85)$$

Then, Eq. (81) decomposes into the coupled equations

$$-i\left[E-\mu+\frac{e^2}{r}\right]F(r) + \frac{dG(r)}{dr} + \frac{k_{(\alpha)}}{r}G(r) = 0, \qquad (86)$$

and

$$-i\left[E+\mu+\frac{e^2}{r}\right]G(r) + \frac{dF(r)}{dr} - \frac{k_{(\alpha)}}{r}F(r) = 0, \qquad (87)$$

where we used the fact that $A_0 = -e/r$.

The solutions of these equations are given in terms of the confluent hypergeometric function $M(a,b;x)$ as

$$F(r) = -i\sqrt{\frac{Q}{T}}\frac{e^{-rD}}{2}(rD)^{\gamma_{(\alpha)}-1}\left[M\left(\gamma_{(\alpha)}-1+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right) + \frac{\left(\gamma_{(\alpha)}-1+\tilde{P}\right)}{\left(k_{(\alpha)}+\tilde{Q}\right)}M\left(\gamma_{(\alpha)}+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right], \qquad (88)$$

and

$$G(r) = \frac{e^{-rD}}{2}(rD)^{\gamma_{(\alpha)}-1}\left[M\left(\gamma_{(\alpha)}-1+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right) - \frac{\left(\gamma_{(\alpha)}-1+\tilde{P}\right)}{\left(k_{(\alpha)}+\tilde{Q}\right)}M\left(\gamma_{(\alpha)}+\tilde{P},2\gamma_{(\alpha)}-1;2rD\right)\right], \qquad (89)$$

where $T = \mu - E$; $Q = \mu + E$, $D = \sqrt{TQ} = \sqrt{\mu^2 - E^2}$; $\gamma_{(\alpha)} = 1 + \sqrt{k^2_{(\alpha)} - \tilde{\alpha}^2}$; $\tilde{P} \equiv \frac{\tilde{\alpha}}{2}\left(\sqrt{T/Q} - \sqrt{Q/T}\right)$; $\tilde{Q} \equiv \frac{\tilde{\alpha}}{2}\left(\sqrt{T/Q} + \sqrt{Q/T}\right)$, with $\tilde{\alpha} =\approx \frac{1}{137}$ being the fine structure constant.

The solutions given by (88) and (89) are divergent, unless the following condition is fulfilled

$$\gamma_{(\alpha)} - 1 + \tilde{P} = -n;\ n = 0, 1, 2..., \tag{90}$$

which means that

$$\frac{1}{2}\tilde{\alpha}\left(\sqrt{\frac{T}{Q}} - \sqrt{\frac{Q}{T}}\right) = -\left(n + \gamma_{(\alpha)} - 1\right). \tag{91}$$

¿From this equation we may infer that the energy eigenvalues are given by

$$E = \mu\left[1 + \tilde{\alpha}^2\left(n + \left|k_{(\alpha)}\right|\sqrt{1 - \tilde{\alpha}^2 k^{-2}_{(\alpha)}}\right)^{-2}\right]^{-\frac{1}{2}}. \tag{92}$$

This equation exhibits the angle deficit dependence of the energy levels. It is helpful to introduce the quantum number $n_{(\alpha)}$ that corresponds to the principal quantum number of the non-relativistic theory when $\alpha = 1$,

$$n_{(\alpha)} = n + j_{(\alpha)} + \frac{1}{2}. \tag{93}$$

Therefore, Eq. (92) may be cast in the form

$$E_{n_{(\alpha)}, j_{(\alpha)}} = \mu\left\{1 + \tilde{\alpha}^2\left[\left(n_{(\alpha)} - j_{(\alpha)} - \frac{1}{2}\right) + \left(j_{(\alpha)} + \frac{1}{2}\right)\sqrt{1 - \tilde{\alpha}^2\left(j_{(\alpha)} + \frac{1}{2}\right)^{-2}}\right]^{-2}\right\}^{-\frac{1}{2}}. \tag{94}$$

This equation can be written in a way which is better suited to physical interpretation. Thus, as $\tilde{\alpha} \ll 1$, we can expand Eq. (94) in a powers of $\tilde{\alpha}$, and as a result we get the following leading terms

$$E_{n_{(\alpha)}, j_{(\alpha)}} = \mu - \mu\frac{\tilde{\alpha}^2}{2n^2_{(\alpha)}} + \mu\frac{\tilde{\alpha}^4}{2n^4_{(\alpha)}}\left(\frac{3}{4} - \frac{n_{(\alpha)}}{j_{(\alpha)} + \frac{1}{2}}\right). \tag{95}$$

The first term corresponds to the rest energy of the electron; the second one gives the energy of the bound states in the non-relativistic approximation and the third one corresponds to the relativistic correction. Note that these last two terms depend on the angle deficit. The further terms can be neglected in comparison with these first three terms.

Now, let us consider the total shift in the energy between the states with $j = n - \frac{1}{2}$, and $j = \frac{1}{2}$, for a given n. This shift is given by

$$\Delta E_{n_{(\alpha)}, j_{(\alpha)}} = \frac{\mu e^8}{n^3_{(\alpha)}}\left(\frac{n_{(\alpha)} - 1}{2\left[n_{(\alpha)} + m\left(\frac{1}{\alpha} - 1\right)\right]\left[1 + m\left(\frac{1}{\alpha} - 1\right)\right]}\right). \tag{96}$$

One important characteristic of Eq. (94) is that it contains a dependence on n, j and α. The dependence on α corresponds to an analogue of the electromagnetic Aharonov-Bohm effect for bound states, but now in the gravitational context. Therefore, the interaction with the topology(conical singularity) causes the energy levels to change. Note that the presence of the cosmic string destroys the degeneracy of all the levels, corresponding to $l = 0$ and $l = 1$, and destroys partially this degeneracy for the other sub-levels. Therefore, as the occurrence of degeneracy can often be ascribed to some symmetry property of the physical system, the fact that the presence of the cosmic string destroys the degeneracy means that there is a break of the original symmetry. Observe that for $\alpha = 1$, the results reduce to the flat Minkowski space-time case as expected.

As a estimation of the effect of the cosmic string on the energy shift of the hydrogen atom, let us consider $\alpha = 1 - 10^{-6}$ which corresponds to GUT cosmic strings. Using this value into Eq. (96), we conclude that the presence of the cosmic string reduces the energy of the level of the states $2P_{1/2}(n = 2$, $l = 1$, $j = l - \frac{1}{2} = \frac{1}{2}$, $m = 1)$ to about 10^{-4}% in comparison with the flat space-time value. This decrease is of the order of the measurable Zeeman effect in carbon atoms for $2P$ states when submitted, for example, to an external magnetic field with strength to about tens of Tesla. Therefore, this shift in the energy levels produced by a cosmic string is measurable as well.

Finally, we can write down the general solution to Eq. (70) corresponding to a hydrogen atom placed in the background space-time of a cosmic string. Thus, it reads the

$$\Psi_{l_{(\alpha)},j_{(\alpha)}=l_{(\alpha)}+\frac{1}{2},m_{(\alpha)}}(x) = e^{-iEt}r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}(\sin\theta)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \times F_{(\alpha)}(r)\begin{pmatrix} \sqrt{\frac{l_{(\alpha)}+m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}Y_{l(\alpha)}^{m_{(\alpha)}-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l_{(\alpha)}-m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}Y_{l(\alpha)}^{m_{(\alpha)}+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \tag{97}$$

and

$$\Psi_{l_{(\alpha)},j_{(\alpha)}=l_{(\alpha)}-\frac{1}{2},m_{(\alpha)}}(x) = e^{-iEt}r^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)}(\sin\theta)^{-\frac{1}{2}\left(1-\frac{1}{\alpha}\right)} \times G_{(\alpha)}(r)\begin{pmatrix} -\sqrt{\frac{l_{(\alpha)}-m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}Y_{l(\alpha)}^{m_{(\alpha)}-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l_{(\alpha)}+m_{(\alpha)}+\frac{1}{2}}{2l_{(\alpha)}+1}}Y_{l(\alpha)}^{m_{(\alpha)}+\frac{1}{2}}(\theta,\phi) \end{pmatrix}, \tag{98}$$

where $F_{(\alpha)}(r)$ and $G_{(\alpha)}(r)$ are given by Eqs. (88) and (89), respectively, and the index α was introduced to emphasize the dependence of these functions on this parameter.

Note that the solutions depend on the topological features of the space-time of a cosmic string whose influence appears codified in the parameter α associated with the presence of the cosmic string and this is the point at issue here.

B. Kratzer potential in the space-time of a global monopole

In order to do these studies let us consider that a non-relativistic particle living in a curved space-time is described by the Schrödinger equation which should take the form

$$i\frac{\partial\psi}{\partial t} = -\frac{1}{2\mu}\nabla^2_{LB}\psi + V\psi, \tag{99}$$

where ∇^2_{LB} is the Laplace-Beltrami operator, the covariant version of the Laplacian given by $\nabla^2_{LB} = g^{-\frac{1}{2}}\partial_i\left(g^{ij}g^{\frac{1}{2}}\partial_j\right)$, with $i,j = 1,2,3$; $g = \det(g_{ij})$; μ is the mass of the particle and V is an external potential.

Now, let us consider a particle placed in the space-time of a global monopole, interacting with a Kratzer potential given by

$$V(r) = -2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right), \tag{100}$$

where A and D are positive constants.

In order to determine the energy spectrum let us write the Schrödinger equation in the background space-time of a global monopole. Then, we get

$$-\frac{1}{2\mu b^2 r^2}\left[2rb^2\frac{\partial}{\partial r} + b^2r^2\frac{\partial^2}{\partial r^2} - \mathbf{L}^2 - 2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right)\right]\psi(\mathbf{r}) = E\psi(\mathbf{r}), \tag{101}$$

where $\mathbf{L}$ is the usual orbital angular momentum operator. We begin by using the standard procedure for solving Eq. (101) and assume that the eigenfunction can be written as

$$\psi_{m,l}(\mathbf{r}) = R_l(r)Y_l^m(\theta,\varphi). \tag{102}$$

Substituting Eq.(102) into Eq.(101), we get

$$-\frac{1}{2\mu}\frac{d^2g_l(r)}{dr^2} - 2D\left(\frac{A}{r} - \frac{1}{2}\frac{A^2}{r^2}\right)g_l(r) + \frac{1}{2\mu}\frac{l(l+1)}{b^2r^2}g_l(r) = Eg_l(r), \tag{103}$$

where $g_l(r) = rR_l(r)$.

The solution of Eq.(103) can be written as

$$g_l(r) = r^{\lambda_l}e^{-\bar{\beta}r}F_l(r) \tag{104}$$

where

$$\lambda_l = \frac{1}{2} + \frac{1}{2}\sqrt{1+4\left(2\mu DA^2 + \frac{l(l+1)}{b^2}\right)}, \tag{105}$$

and

$$\bar{\beta}^2 = -2\mu E > 0. \tag{106}$$

Substituting Eq. (104) into Eq. (103) and making use of Eqs. (105) and (106) we obtain the equation for $F(z)$

$$z\frac{d^2F(z)}{dz^2} + (2\lambda_l - z)\frac{dF(z)}{dz} - \left(\lambda_l - \frac{2mAD}{\bar{\beta}}\right)F(z) = 0, \tag{107}$$

where $z = 2\bar{\beta}r$.

The solution of this equation is the confluent hypergeometric function ${}_1F_1\left(\lambda_l - \frac{\gamma^2}{\bar{\beta}A}, 2\lambda_l \,;\, 2\bar{\beta}r\right)$, with $\gamma^2 = 2\mu DA^2$.

Therefore, the solution for the radial function $g_l(r)$ is given by

$$g_l(r) = r^{\lambda_l}e^{-\bar{\beta}r}{}_1F_1\left(\lambda_l - \frac{\gamma^2}{\bar{\beta}A}, 2\lambda_l \,;\, 2\bar{\beta}r\right). \tag{108}$$

In order to make $g_l(r)$ vanishes for $r \to \infty$, the confluent hypergeometric function may increase not faster than some power of r, that is, the function must be a polynomial. Hence

$$\lambda_l - \frac{\gamma^2}{\bar{\beta}A} = -\bar{n}_r,\ \bar{n}_r = 0,1,2,\ldots. \tag{109}$$

With this condition we find that the eigenvalues are given by

$$E_{l,\bar{n}_r} = -\frac{1}{2\mu A^2}\gamma^4\left(\bar{n}_r + \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{l(l+1)}{b^2} + \gamma^2}\right)^{-2} \tag{110}$$

It is worth noticing from expression for the energy given by Eq. (110) that even in the case in which the $z-$component of the angular momentum vanishes the energy level is shifted relative to the Minkowski case.

As an estimation of the effect of the global monopole on the energy spectrum, let us consider a stable global monopole configuration for which $\eta = 0.19m_p$, where m_p is the Planck mass. In this situation the shift in the energy spectrum between the first two levels in this space-time decreases of about 82% as compared with the Minkowski space-time. For symmetry breaking at grand unification scale, the typical value of $8\pi G\eta^2$ is around 10^{-6} and in this case the energy shift decreases of about 1%.

C. Harmonic oscillator in the space-time of Safko-Witten

Let us consider the line element corresponding to Safko-Witten space-time which is given by Eq.(7). The Schrödinger equation in this space-time reads as

$$-\frac{1}{2\mu}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\rho + \frac{1}{e^{-2\beta}\rho^2}\partial_\theta^2 + \partial_z^2\right]\psi(t,\rho,\theta,z)$$

$$+V(\rho,z)\psi(t,\rho,\theta,z) = i\frac{\partial}{\partial t}\psi(t,\rho,\theta,z), \tag{111}$$

where $V(\rho,z)$ is the interaction potential corresponding to a three-dimensional harmonic oscillator which is assumed to be

$$V(\rho,z) = \frac{1}{2}\mu w^2\left(\rho^2 + z^2\right). \tag{112}$$

We will now determine the eigenfunction of the Eq.(111), with the interaction potential given by Eq.(112), by searching for solutions of the form

$$\psi(t,\rho,\theta,z) = \frac{1}{\sqrt{2\pi}}e^{-iEt+im\theta}R(\rho)Z(z). \tag{113}$$

Equation (111) leads to two ordinary differential equations for $R(\rho)$ and $Z(z)$ which are given by

$$-\frac{1}{2\mu}\left[\frac{1}{R(\rho)}\frac{d^2R(\rho)}{d\rho^2} + \frac{1}{R(\rho)\rho}\frac{dR(\rho)}{d\rho} - \frac{m^2}{e^{-2\beta}\rho^2}\right] + \frac{1}{2}\mu w^2\rho^2 = \Omega \tag{114}$$

and

$$-\frac{1}{2\mu Z(z)}\frac{d^2Z(z)}{dz^2} + \frac{1}{2}\mu w^2 z^2 = \varepsilon_z, \tag{115}$$

where Ω is a separation constant and such that

$$\Omega + \varepsilon_z = E. \tag{116}$$

Equation (115) is the Schrödinger equation for a particle in the presence of one-dimensional harmonic oscillator potential, and then we have the well-known results

$$\varepsilon_z = \left(n_z + \frac{1}{2}\right)w;\ n_z = 0,1,2,..., \tag{117}$$

with

$$Z(z) = 2^{-\frac{n_z}{2}}\left(n_z!\right)^{-\frac{1}{2}}\left(\frac{\mu w}{\pi}\right)^{\frac{1}{4}}e^{-\frac{\mu w}{2}z^2}H_{n_z}\left(\sqrt{\mu w}z\right), \tag{118}$$

where H_{n_z} is the Hermite Polynomial.

Now, let us look for solutions of Eq. (114). Its solution can be written as

$$R(\rho) = \exp\left(-\frac{\tau}{2}\rho^2\right)\rho^{|m|e^{2\beta}}F(\rho), \tag{119}$$

where $\tau = mw$ and

$$F(\rho) = {}_1F_1\left(\frac{1}{2} + \frac{|m|}{2e^{-\beta}} - \frac{\mu\Omega}{2\tau}, \frac{A}{2}; \tau\rho^2\right) \tag{120}$$

is the degenerate hypergeometric function, with $A = 1 + 2\frac{|m|}{e^{-\beta}}$.

In order to have normalizable wave-function, the series in Eq. (120) must be a polynomial of degree n_ρ, and therefore

$$\frac{1}{2} + \frac{|m|}{2e^{-2\beta}} - \frac{\mu\Omega}{2\tau} = -n_\rho;\ n_\rho = 0,1,2,.... \tag{121}$$

With this condition, we obtain the following energy eigenvalues

$$\Omega = w\left(1+\frac{|m|}{e^{-2\beta}}+2n_\rho\right). \tag{122}$$

If we substitute Eqs. (122) and (117) into (116) we get, finally, the energy eigenvalues

$$E_N = w\left(N+\frac{|m|}{e^{-2\beta}}+\frac{3}{2}\right), \tag{123}$$

where $N = 2n_\rho + n_z$.

Therefore, the complete eigenfunctions are then given by

$$\begin{aligned}\psi(t,\rho,\theta,z) \;&=\; C_{Nm}e^{-iE_N t}e^{-\frac{\tau}{2}\rho^2}\rho^{|m|e^{2\beta}}F_1\left(\frac{1}{2}+\frac{|m|}{2e^{-2\beta}}-\frac{\mu\Omega}{2\tau},\frac{A}{2};\tau\rho^2\right)\\ &\times e^{im\theta}2^{-\frac{n_z}{2}}\left(n_z!\right)^{-\frac{1}{2}}\left(\frac{\mu w}{\pi}\right)^{\frac{1}{4}}e^{-\frac{\mu w}{2}z^2}H_{n_z}\left(\sqrt{\mu w}z\right),\end{aligned} \tag{124}$$

where C_{Nm} is a normalization constant. It is worth calling attention to the fact that the presence of the tubular matter source with an interior magnetic field breaks the degeneracy of the energy levels.

In the case under consideration the shift in the energy spectrum between the first two levels in this background increases of about $10^{-5}\%$ as compared with the flat Minkowski space-time case.

IV. FINAL REMARKS

The loop variables in the space-time of a cosmic string are elements of the Lorentz group. Therefore, for a given curve in this space-time, the phase shift acquired by a vector is an element of the Lorentz group. When a particle is parallel transported along a curve around a cosmic string, it acquires a phase which is different from zero. This fact is a manifestation of the phenomenon called gravitational Aharonov-Bohm effect, which in this case, differently from the electromagnetic case, appears at purely classical level.

The radiation emitted by a scalar particle moving along a geodesic line in the point-like global monopole space-time arises due to the geometric and topological features of this space-time. Considering the case of a scalar field minimally coupled with gravity and a specific situation in which the solid angle deficit is small we find that the total energy radiated by a particle along its trajectory is proportional to the cube of the velocity and to the cube of the Lorenz parameter in the non-relativistic and ultra-relativistic cases, respectively.

As a conclusion we can say that particles moving along geodesic lines in the space-time of a point-like global monopole will emit radiation in the same way as in case of an infinitely thin cosmic string space-time [6]. Analogously to the case of an infinitely thin cosmic string space-time, the energy emitted depends on the angle deficit and vanishes when this angle deficit vanishes, but in the present case, this radiation arises associated with the curvature and non-trivial topology of the space-time of the global monopole, differently from the cosmic string case in which the effect comes exclusively from the non-trivial topology of the space-time.

If a charged particle is placed in the space-time of Safko-Witten, it experiences an electrostatic self-force associated with the conical structure of the background space-time. This structure deforms the electrostatic field of the particle and this deformation depends on the distance between the particles and the tube and on the magnetic field in the interior of the tube, in such a way that an electrostatic self-force appears.

The presence of a cosmic string changes the solution and shifts the energy levels of a hydrogen atom as compared with the flat Minkowski space-time result. It is interesting to observe that these shifts depend on the parameter that defines the angle deficit and vanish when the angle deficit vanishes. These shifts arise from the topological features of the space-time generated by this defect.

The shifts in the energy is only two orders of magnitude less than the ratio between the fine structure splitting and the energy of the ground state of the non-relativistic hydrogen atom and is of the order of the Zeeman effect. Therefore, the modifications in the spectra of the hydrogen atom due to the presence of the gravitational field of a string are all measurable, in principle.

In the space-time of a global monopole the quantum dynamics a particle interacting with a Kratzer potential depends on the geometric and topological features of this space-time. The presence of the defect shifts the energy levels as compared to the flat Minkowski space-time one. It is interesting to observe that these shifts depend on the parameter that defines the solid angle deficit. Also the dynamics of a non-relativistic quantum oscillator depends on the topological features of this space-time through the angle deficit associated with its geometry.

In the case of the harmonic oscillator in Safko-Witten space-time, the wavefunction as well as the energy levels are modified by the conical structure associated with this space-time, which is due to the combined effect of the distribution of matter and the interior magnetic field.

The obtained results show how the geometry and a nontrivial topology influences the energy spectrum as compared with the flat space-time case and show how these quantities depend on the surroundings and their characteristics.

Therefore, the problem of finding how the energy spectrum of an atom placed in a gravitational field is perturbed by this background has to take into account not only the geometrical, but also the topological features of the space-times under consideration. In other words, the behavior of an atomic system is determined not only by the curvature at the position of the atom, but also by the topology of the background space-time.

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