Classification of Energy Momentum Tensors in $n \geq 5$ Dimensional Space-times: A Review

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Recent developments in string theory suggest that there might exist extra spatial dimensions, which are not small nor compact. The framework of a great number of brane cosmological models is that in which the matter fields are confined on a brane-world embedded in five dimensions (the bulk). Motivated by this we review the main results on the algebraic classification of second order symmetric tensors in 5-dimensional space-times. All possible Segre types for a symmetric two-tensor are found, and a set of canonical forms for each Segre type is obtained. A limiting diagram for the Segre types of these symmetric tensors in 5–D is built. Two theorems which collect together some basic results on the algebraic structure of second order symmetric tensors in 5–D are presented. We also show how one can obtain, by induction, the classification and the canonical forms of a symmetric two-tensor on n-dimensional ($n > 5$) spaces from its classification in 5–D spaces, present the Segre types in n–D and the corresponding canonical forms. This classification of symmetric two-tensors in any n–D spaces and their canonical forms are important in the context of n-dimensional brane-worlds context and also in the framework of 11–D supergravity and 10–D superstrings.

1   Introduction

Recent developments in string theory and its extension M-theory have suggested a scenario in which the matter fields are confined on 3–D brane-world embedded in $1 + 3 + d$ dimensions (the bulk). It is not necessary for the d extra spatial dimensions to be small and compact, which is a fundamental departure from the standard Kaluza-Klein unification scheme. Within the brane-world scenario only gravity and other exotic matter such as the dilaton can propagate in the $(1 + 3 + d)$–dimensional bulk. Furthermore, in general the number d of extra dimensions is not necessarily equal to 1. However, this general paradigm is often simplified to a 5–D context, where matter fields are restricted to a 4–D space-time while gravity acts in 5–D [1, 2]. In this *limited* 5–D framework a great number work in *brane-world cosmology* has been done (see, for example, [3, 4] and references therein). Motivated by this scenario, in this article we present a review of our studies on the algebraic classification, limits of Segre types and algebraic structures of second order symmetric tensors (energy-momentum, Einstein and Ricci tensors) in 5–D and n–D space-times [5] – [8].

It is well known that a coordinate-invariant characterization of the gravitational field in general relativity (GR) is given in terms of the curvature tensor and a finite number of its covariant derivatives relative to canonically chosen Lorentz frames [9] – [11]. The Riemann tensor itself is decomposable into three irreducible parts, namely the Weyl tensor (denoted by W_{abcd}), the traceless Ricci tensor

$(S_{ab} \equiv R_{ab} - \frac{1}{4} R \, g_{ab})$ and the Ricci scalar $(R \equiv R_{ab} \, g^{ab})$. The algebraic classification of the Weyl part of the Riemann tensor, known as Petrov classification, has played a significant role in the study of various topics in general relativity. However, for full classification of curvature tensor of nonvacuum space-times one also has to consider the Ricci part of the curvature tensor, which by virtue of Einstein's equations $G_{ab} = \kappa \, \mathcal{T}_{ab} + \Lambda \, g_{ab}$ clearly has the same algebraic structure of both the Einstein tensor $G_{ab} \equiv R_{ab} - \frac{1}{2} R \, g_{ab}$ and the energy momentum tensor $\mathcal{T}_{ab}$.

The algebraic classification of a symmetric two-tensor locally defined on a 4-dimensional Lorentzian manifold (such as the Ricci, Einstein and energy momentum tensors) is known as Segre classification, and has been discussed by several authors [12]. It is of interest in at least three contexts: one is in understanding some purely geometrical features of space-times [13] – [15]. The second one is in classifying and interpreting matter field distributions [16] – [24]. The third is as part of the procedure for checking whether apparently different space-times are in fact locally the same up to coordinate transformations [9] – [11]. For examples of the use of this invariant characterization in a class of Gödel-type space-times [25] see [26].

Before presenting the scope of this article, for the sake of completeness, let us briefly recall some historical points, and in the next two paragraphs we mention, in passing, other contexts where the results of the present paper may also be of some interest.

The idea that the various interactions in nature might be

unified by enlarging the dimensionality of the space-time has a long history that goes back to the works of Nordström, Kaluza and Klein [27, 29]. Its earlier adherents were mainly interested in extending general relativity, but a late increased interest has been apparent in the particle physics community, especially among those investigating super-symmetry.

An approach to the 5–D non-compact Kaluza-Klein scenario, known as space-time-matter (STM) theory, has also been discussed in a number of papers (see, e.g., [30] – [33] and references therein; and also [8], [34] – [36]).

In this paper we review our main results on the algebraic classification, limits and their relation to Segre types in 5–D, and algebraic properties of second order symmetric tensors (energy momentum, Einstein and Ricci) in 5–D space-times [5] – [8]. We note that the proof of Theorem 1, the consequent treatment of the classification in Segre types and the corresponding canonical forms presented here are new. We also show how one can obtain, by induction, the classification and the canonical forms of a symmetric two-tensor on n-dimensional ($n > 5$) spaces from its classification in 5-dimensional spaces. This classification of symmetric two-tensors in any n-dimensional spaces and their canonical forms are important in the context of n-dimensional brane-worlds context as well as in the framework of 11–D supergravity and 10–D superstrings.

To make this paper as clear and selfcontained as possible, in the next section we introduce the notation and the underlying mathematical setting, which will be used throughout the article. In section 3 we present a new proof of a theorem previously stated in [7] (see Lemma 3.1). Using this theorem (Theorem 1) and the classification of an arbitrary symmetric two-tensors S defined on a 4–D space-time, we derive a classification of any second order symmetric tensors in 5–D, and obtain a set of canonical forms for each class. In section 4 we recall the concept of limits of space-time, hereditary properties, and build a limiting diagram for the Segre types in 5–D. In section 5 we show how one can obtain, by induction, the algebraic classification and the canonical forms of symmetric two-tensors on a n-dimensional space (general arena of brane-worlds theories) from their classification in 5–D. In section 6 we collect together in two theorems some basic results regarding eigenvectors and invariant subspaces of $R^a_{\ b}$, which generalize theorems on 4–D space-times [12, 37]. In the last section we present some concluding remarks.

Although the Ricci and the energy momentum tensors are constantly referred to, our results apply to any second order real symmetric tensor defined on 5–D (and sometime n–D) Lorentzian manifolds.

2 Prerequisites

In this section we shall fix our general setting, define the notation and briefly recall some important results required for the remainder of this work.

The algebraic classification of the Ricci tensor at a point $p \in M$ can be cast in terms of the eigenvalue problem

$$(R^a_{\ b} - \lambda\, \delta^a_b)\, V^b = 0 \, , \qquad (2.1)$$

where λ is a scalar, V^b is a vector and the mixed Ricci tensor $R^a_{\ b}$ may be thought of as a linear operator $R : T_p(M) \longrightarrow T_p(M)$. In this work M is a real 5-dimensional (or n-dimensional, in section 5) space-time manifold locally endowed with a Lorentzian metric of signature $(-++++)$, [or $(-+\cdots+)$ in n–D case] $T_p(M)$ denotes the tangent space to M at a point $p \in M$ and Latin indices range from 0 to 4, unless otherwise stated. Although the matrix $R^a_{\ b}$ is real, the eigenvalues λ and the eigenvectors V^b are often complex. A mathematical procedure used to classify matrices in such a case is to reduce them through similarity transformations to canonical forms over the complex field. Among the existing canonical forms the Jordan canonical form (JCF) turns out to be the most appropriate for a classification of $R^a_{\ b}$.

A basic result in the theory of JCF [38] is that given an n-square matrix R over the complex field, there exist nonsingular matrices X such that

$$X^{-1} R X = J \, , \qquad (2.2)$$

where J, the JCF of R, is a block diagonal matrix, each block being of the form

$$J_r(\lambda_k) = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda_k \end{bmatrix} . \qquad (2.3)$$

Here r is the dimension of the block and λ_k is the k-th root of the characteristic equation $\det(R - \lambda I) = 0$. Hereafter R will be the real matrix formed with the mixed components $R^a_{\ b}$ of the Ricci tensor.

A Jordan matrix J is uniquely defined up to the ordering of the Jordan blocks. Further, regardless of the dimension of a specific Jordan block there is one and only one independent eigenvector associated to it.

In the Jordan classification two square matrices are said to be equivalent if similarity transformations exist such that they can be brought to the same JCF. The JCF of a matrix gives explicitly its eigenvalues and makes apparent the dimensions of the Jordan blocks. However, for many purposes a somehow coarser classification of a matrix is sufficient. In the Segre classification, for example, the value of the roots of the characteristic equation is not relevant — only the dimension of the Jordan blocks and the degeneracies of the eigenvalues matter. The Segre type is a list $[r_1 r_2 \cdots r_m]$ of the dimensions of the Jordan blocks. Equal eigenvalues in distinct blocks are indicated by enclosing the corresponding digits inside round brackets. Thus, for example, in the degenerated Segre type $[(21)11]$ three out of the five eigenvalues are equal; there are four linearly independent eigenvectors, two of which are associated to Jordan blocks of dimensions 2 and 1, and the last eigenvectors corresponds to two blocks of dimension 1.

In classifying symmetric tensors in a Lorentzian space two refinements to the usual Segre notation are often used. Instead of a digit to denote the dimension of a block with

complex eigenvalue a letter is used, and the digit corresponding to a time-like eigenvector is separated from the others by a comma.

In this work we shall deal with two types of pentad of vectors, namely the semi-null pentad basis $\{\mathbf{l}, \mathbf{m}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$, whose non-vanishing inner products are only

$$l^a m_a = x^a x_a = y^a y_a = z^a z_a = 1 \, , \qquad (2.4)$$

and the Lorentz pentad basis $\{\mathbf{t}, \mathbf{w}, \mathbf{x}, \mathbf{y}, \mathbf{z}\}$, whose only non-zero inner products are

$$-t^a t_a = w^a w_a = x^a x_a = y^a y_a = z^a z_a = 1 \, . \qquad (2.5)$$

At a point $p \in M$ the most general decomposition of R_{ab} in terms of a Lorentz basis for symmetric tensors at $p \in M$ is manifestly given by

$$
\begin{aligned}
R_{ab} = {} & \sigma_1 \, t_a t_b + \sigma_2 \, w_a w_b + \sigma_3 \, x_a x_b + \sigma_4 \, y_a y_b + \sigma_5 \, z_a z_b + 2\,\sigma_6 \, t_{(a} w_{b)} \\
& + 2\,\sigma_7 \, t_{(a} x_{b)} + 2\,\sigma_8 \, t_{(a} y_{b)} + 2\,\sigma_9 \, t_{(a} z_{b)} + 2\,\sigma_{10} \, w_{(a} x_{b)} + 2\,\sigma_{11} \, w_{(a} y_{b)} \\
& + 2\,\sigma_{12} \, w_{(a} z_{b)} + 2\,\sigma_{13} \, x_{(a} y_{b)} + 2\,\sigma_{14} \, x_{(a} z_{b)} + 2\,\sigma_{15} \, y_{(a} z_{b)} \, ,
\end{aligned}
\qquad (2.6)
$$

where the coefficients $\sigma_1, \ldots, \sigma_{15} \in \mathbb{R}$.

3   Classification in 5–D

In this section we shall present the algebraic classification of symmetric two-tensors defined on 5–D space-times, which is based upon two ingredients: one is the classification of second order symmetric tensors in 4–D, and the other is a theorem stated below (proved in [7]; see Lemma 3.1). We emphasize, however, the proof presented here is completely new.

Theorem 1 *Let M be a real 5-dimensional manifold endowed with a Lorentzian metric g of signature $(-++++)$. Let R^a_b be the mixed form of a second order symmetric tensor R defined at any point $p \in M$. Then R^a_b has at least one real non-null eigenvector with real eigenvalue.*

Proof. We initially consider the cases when all eigenvalues of R are real. Suppose first that R^a_b has a single eigenvector. According to the above results and eq. (2.2) it can be brought to a Jordan canonical form J^a_b with only one Jordan block, namely

$$
J^a_b = \begin{bmatrix}
\lambda & 1 & 0 & 0 & 0 \\
0 & \lambda & 1 & 0 & 0 \\
0 & 0 & \lambda & 1 & 0 \\
0 & 0 & 0 & \lambda & 1 \\
0 & 0 & 0 & 0 & \lambda
\end{bmatrix} \, , \qquad (3.1)
$$

where $\lambda \in \mathbb{R}$. Clearly in this case the Segre type for R^a_b is [5]. The matricial equation (2.2) implies that $R\,X = J\,X$. Using J given by (3.1) and equating the corresponding columns of both sides of this equation one obtains

$$
\begin{aligned}
R\,\mathbf{X_1} &= \lambda \mathbf{X_1} \, , & (3.2) \\
R\,\mathbf{X_2} &= \lambda \mathbf{X_2} + \mathbf{X_1} \, , & (3.3) \\
R\,\mathbf{X_3} &= \lambda \mathbf{X_3} + \mathbf{X_2} \, , & (3.4) \\
R\,\mathbf{X_4} &= \lambda \mathbf{X_4} + \mathbf{X_3} \, , & (3.5) \\
R\,\mathbf{X_5} &= \lambda \mathbf{X_5} + \mathbf{X_4} \, , & (3.6)
\end{aligned}
$$

where we have denoted by $\mathbf{X_A}$ ($\mathbf{A} = 1, \cdots, 5$) the column vectors of the matrix X. Clearly $\mathbf{X_1}$ is an eigenvector of R. Since R is a symmetric two-tensor, from equations (3.2) and (3.3) one finds that $\mathbf{X_1}$ is a null vector. Thus, if R^a_b admits a single eigenvector it must be null. However, the Lorentzian character of the metric g on M, together with the symmetry of R_{ab}, rule out this Jordan canonical form for R^a_b. Indeed, from equations (3.2), (3.4) and (3.5) one can easily show that $\mathbf{X_1}.\mathbf{X_2} = \mathbf{X_1}.\mathbf{X_3} = 0$, which together with eqs. (3.3) and (3.4) imply $\mathbf{X_2}.\mathbf{X_2} = 0$. Thus, $\mathbf{X_1}$ and $\mathbf{X_2}$ are both null and orthogonal one to another. As the metric g is Lorentzian they must be proportional, hence X is a singular matrix. So, at a point $p \in M$ the Ricci tensor R cannot be of Segre type [5], it cannot have a single eigenvector.

Suppose now that R^a_b has two linearly independent eigenvectors $(\mathbf{k}, \mathbf{n})$ and that they are both null vectors. Let μ and ν be the associated eigenvalues. Then

$$
\begin{aligned}
R^a_b \, k^b &= \mu \, k^a \, , & (3.7) \\
R^a_b \, n^b &= \nu \, n^a \, , & (3.8)
\end{aligned}
$$

where $\mu, \nu \in \mathbb{R}$. Since R_{ab} is symmetric and $(\mathbf{k}, \mathbf{n})$ are linearly independent, eqs. (3.7) and (3.8) imply that $\mu = \nu$. Thus, from eqs. (3.7) and (3.8) the vector $\mathbf{v} = \mathbf{k} + \mathbf{n}$ is also an eigenvector of R with real eigenvalue μ. Since $\mathbf{k}$ and $\mathbf{n}$ are linearly independent null vectors it follows that $v^a v_a = 2\,k^a n_a \neq 0$, so $\mathbf{v}$ is a non-null eigenvector with real eigenvalue.

In the cases where R^a_b has more than two linearly independent null eigenvectors one can always use two of them to similarly construct one non-null eigenvector with real eigenvalue. Thus, when all eigenvalues of R are real R^a_b has at least one non-null eigenvector with real eigenvalue.

The case when the Ricci tensor has complex eigenvalues can be dealt with by using an approach borrowed from [12] as follows. Suppose that $\alpha \pm i\beta$ are complex eigenvalues of R^a_b corresponding to the eigenvectors $\mathbf{V}_\pm = \mathbf{Y} \pm i\mathbf{Z}$, where α and $\beta \neq 0$ are real and $\mathbf{Y}$, $\mathbf{Z}$ are independent vectors defined on $T_p(M)$. Since R_{ab} is symmetric and the eigenvalues are different, the eigenvectors must be orthogonal and

hence equation $\mathbf{Y}.\mathbf{Y} + \mathbf{Z}.\mathbf{Z} = 0$ holds. It follows that either one of the vectors $\mathbf{Y}$ or $\mathbf{Z}$ is time-like and the other space-like or both are null and, since $\beta \neq 0$, not collinear. Regardless of whether $\mathbf{Y}$ and $\mathbf{Z}$ are both null vectors or one time-like and the other space-like, the real and the imaginary part of (2.1) give

$$R^a_{\ b} Y^b = \alpha Y^a - \beta Z^a , \qquad (3.9)$$
$$R^a_{\ b} Z^b = \beta Y^a + \alpha Z^a . \qquad (3.10)$$

Thus, the vectors $\mathbf{Y}$ and $\mathbf{Z}$ span a time-like 2-dimensional subspace of $T_p(M)$ invariant under $R^a_{\ b}$. Besides, by a similar procedure to that used in [5] one can show that the 3-dimensional space orthogonal to this time-like 2-space is space-like, is also invariant under $R^a_{\ b}$ and contains three orthogonal eigenvectors of $R^a_{\ b}$ with real eigenvalues. Thus, when $R^a_{\ b}$ has complex eigenvalues we again have at least

one non-null eigenvector (actually we have at least three) with real eigenvalue. This completes the proof of Theorem 1.

We shall now discuss the algebraic classification of the Ricci tensor. Let $\mathbf{v}$ be the real non-null eigenvector of Theorem 1 and let $\eta \in \mathbb{R}$ be the corresponding eigenvalue. Obviously the normalized vector $\mathbf{u}$ defined by

$$u^a = \frac{v^a}{\sqrt{\epsilon\, v^a v_a}} \qquad \text{with} \qquad \epsilon \equiv \text{sign}\,(v^a v_a) = u^a u_a \qquad (3.11)$$

is also an eigenvector of $R^a_{\ b}$ associated to η.

If the vector $\mathbf{u}$ is time-like ($\epsilon = -1$), one can choose it as the time-like vector $\mathbf{t}$ of a Lorentz pentad $\{\mathbf{t}, \tilde{\mathbf{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}}\}$. The fact that $\mathbf{u} \equiv \mathbf{t}$ is an eigenvector of $R^a_{\ b}$ can then be used to reduce the general decomposition (2.6) to

$$\begin{aligned}
R_{ab} =\ & -\eta\, t_a t_b + \sigma_2\, \tilde{w}_a \tilde{w}_b + \sigma_3\, \tilde{x}_a \tilde{x}_b + \sigma_4\, \tilde{y}_a \tilde{y}_b + \sigma_5\, \tilde{z}_a \tilde{z}_b + 2\,\sigma_{10}\, \tilde{w}_{(a} \tilde{x}_{b)} \\
& + 2\,\sigma_{11}\, \tilde{w}_{(a} \tilde{y}_{b)} + 2\,\sigma_{12}\, \tilde{w}_{(a} \tilde{z}_{b)} + 2\,\sigma_{13}\, \tilde{x}_{(a} \tilde{y}_{b)} + 2\,\sigma_{14}\, \tilde{x}_{(a} \tilde{z}_{b)} \\
& + 2\,\sigma_{15}\, \tilde{y}_{(a} \tilde{z}_{b)} ,
\end{aligned} \qquad (3.12)$$

where $\eta = -\sigma_1$. Using (2.5) and (3.12) one finds that the mixed matrix $R^a_{\ b}$ takes the block diagonal form

$$R^a_{\ b} = S^a_{\ b} - \eta\, t^a t_b . \qquad (3.13)$$

The first block is a (4×4) symmetric matrix acting on the 4–D space-like vector space orthogonal to the subspace $\mathcal{U}$ of $T_p(M)$ defined by $\mathbf{u}$. Hence it can be diagonalized by spatial rotation of the basis vectors $(\tilde{\mathbf{w}}, \tilde{\mathbf{x}}, \tilde{\mathbf{y}}, \tilde{\mathbf{z}})$. The second block is 1-dimensional and acts on the subspace $\mathcal{U}$. Thus, there exists a Lorentz pentad relative to which $R^a_{\ b}$ takes a diagonal form with real coefficients. The Segre type for $R^a_{\ b}$ is then $[1, 1111]$ or one of its degeneracies.

If $\mathbf{u}$ is space-like ($\epsilon = 1$) one can choose it as the space-like vector $\mathbf{z}$ of a Lorentz pentad and using eqs. (2.5) one similarly finds that $R^a_{\ b}$ takes the block diagonal form

$$R^a_{\ b} = S^a_{\ b} + \eta\, z^a z_b , \qquad (3.14)$$

where $\eta = \sigma_5$. But now the 4–D vector space orthogonal to the space-like subspace of $T_p(M)$ defined by $\mathbf{u}$ is Lorentzian. Then the mixed matrix $S^a_{\ b}$ effectively acts on a 4–D Lorentzian vector space and is not necessarily symmetric, it is not diagonalizable in general. As S_{ab} is obviously symmetric, from equation (3.14) it follows that the algebraic classification of $R^a_{\ b}$ and a set of canonical forms for R_{ab} can be achieved from the classification of a symmetric two-tensor S on a 4–D space-time.

Thus, using the known classification [12] for 4–D space-times it follows that semi-null pentad bases can be introduced at $p \in M$ such that the possible Segre types and the corresponding canonical forms for R are given by

Segre type		Canonical form	
$[1, 1111]$	$R_{ab} =$	$2\,\rho_1\, l_{(a} m_{b)} + \rho_2\, (l_a l_b + m_a m_b) + \rho_3\, x_a x_b + \rho_4\, y_a y_b + \rho_5\, z_a z_b ,$	(3.15)
$[2111]$	$R_{ab} =$	$2\,\rho_1\, l_{(a} m_{b)} \pm l_a l_b + \rho_3\, x_a x_b + \rho_4\, y_a y_b + \rho_5\, z_a z_b ,$	(3.16)
$[311]$	$R_{ab} =$	$2\,\rho_1\, l_{(a} m_{b)} + 2\, l_{(a} x_{b)} + \rho_1\, x_a x_b + \rho_4\, y_a y_b + \rho_5\, z_a z_b ,$	(3.17)
$[z\,\bar{z}111]$	$R_{ab} =$	$2\,\rho_1\, l_{(a} m_{b)} + \rho_2\, (l_a l_b - m_a m_b) + \rho_3\, x_a x_b + \rho_4\, y_a y_b + \rho_5\, z_a z_b ,$	(3.18)

and the twenty-two degeneracies thereof, in agreement with Santos *et al.* [5]. Here $\rho_1, \cdots, \rho_5 \in \mathbb{R}$ and $\rho_2 \neq 0$ in (3.18).

4 Limits and Segre Types in 5–D

We shall use in this section the concept of limit of a space-time introduced in reference [39], wherein by a limit of a space-time, broadly speaking, we mean a limit of a family of space-times as some free parameters are taken to a limit. For instance, in the one-parameter family of Schwarzschild solutions each member is a Schwarzschild space-time with a specific value for the mass parameter m.

In the study of limits of space-times it is worth noticing that there are some properties that are inherited by all limits of a family of space-times [40]. These properties are called hereditary. Thus, for example, a hereditary property devised by Geroch can be stated as follows:

Hereditary property:
Let T be a tensor or scalar field built from the metric and its derivatives. If T is zero for all members of a family of space-times, it is zero for all limits of this family.

$$(4.1)$$

From this property we easily conclude that the vanishing of either the Weyl or Ricci tensor or the curvature scalar are also hereditary properties. In other words, limits of conformally flat space-times are conformally flat, and that limits of Ricci flat space-times are also vacuum solutions.

What can be said about the Petrov and Segre types of those tensors under limiting processes? In general, the algebraic type of the Weyl tensor is not a hereditary property under limiting processes. Nevertheless, to be at least as specialized as the types in the Penrose specialization diagram 4 for the Petrov classification is a hereditary property [40].
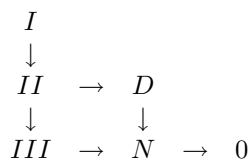
$$
\begin{array}{ccccc}
I & & & & \\
\downarrow & & & & \\
II & \rightarrow & D & & \\
\downarrow & & \downarrow & & \\
III & \rightarrow & N & \rightarrow & 0
\end{array}
$$

Figure 1. Limiting diagram for the Petrov types in 4–D.

For simplicity, in the limiting diagrams in this work, we do not draw arrows between types whenever a compound limit exists. Thus, in figure 4, e.g., the limits $I \rightarrow II \rightarrow D$ imply that the limit $I \rightarrow D$ is allowed.

In 1993 a coordinate-free technique for studying the limits of vacuum space-times was developed and the limits of some well known vacuum solutions were investigated [39]. In that approach the Geroch limiting diagram for the Petrov classification was extensively used. Five years later, to deal with limits on non-vacuum solutions space-times in GR, Paiva *et al.* [41] built a limiting diagram for the Segre types in 4–D. The main results of this paper is that the Segre type of the Ricci tensor is not in general preserved under limiting processes. However, under such processes the Segre types have to be at least as specialized as the types in their diagram for the limits of the Segre type they obtained (see figure 4 of reference [41]).

The characteristic polynomial associated to the eigenvalue problem (2.1), given by

$$|R^a_{\ b} - \lambda \delta^a_{\ b}| \ , \qquad (4.2)$$

is a polynomial of degree five in λ, and can be always factorized over the complex field as

$$(\lambda - \lambda_1)^{d_1} (\lambda - \lambda_2)^{d_2} \ \cdots \ (\lambda - \lambda_r)^{d_r} \ , \qquad (4.3)$$

where λ_i ($i = 1, 2, \cdots, r$) are the distinct roots of the polynomial (eigenvalues), and d_i the corresponding degeneracies. Clearly $d_1 + \cdots + d_r = 5$. To indicate the characteristic polynomial we shall introduce a new list $\{d_1 \, d_2 \, \cdots \, d_r\}$ of eigenvalues degeneracies, referred to as the type of the characteristic polynomial.

The minimal polynomial can be introduced as follows. Let P be a monic matrix polynomial of degree n in $R^a_{\ b}$, i.e.,

$$P = R^n + c_{n-1} R^{n-1} + c_{n-2} R^{n-2} + \cdots + c_1 R + c_0 \delta \ , \qquad (4.4)$$

where δ is the identity matrix and c_n are, in general, complex numbers. The polynomial P is said to be the minimal polynomial of R if it is the polynomial of lowest degree in R such that $P = 0$. It can be shown that the minimal (monic) polynomial is unique and can be factorized as

$$(R - \lambda_1 \delta)^{m_1} (R - \lambda_2 \delta)^{m_2} \ \cdots \ (R - \lambda_r \delta)^{m_r} \ , \qquad (4.5)$$

where m_i is the dimension of the Jordan block of *highest dimension* for each eigenvalue $\lambda_1, \ \lambda_2, \ \cdots \lambda_r$, respectively. We shall denote the minimal polynomial through a third list $\|m_1 \, m_2 \, \cdots \, m_r \|$, called as the type of the minimal polynomial.

Clearly the characteristic (4.2) and the minimal (4.5) polynomials of $R^a_{\ b}$ as well as the eigenvalues are built from the metric and its derivatives [41]. Since they are either scalars or tensors built from the metric and its derivatives, the hereditary property (4.1) can be applied to them.

Thus, limiting diagrams for the types of the five degree characteristic polynomial corresponding to the eigenvalue problem (2.1), and for the types of the minimal polynomial of $R^a_{\ b}$ can be constructed using the hereditary property (4.1). For more details about this point see [42]. From each of these limiting diagrams (see figures 1 and 2 of [42]) one can construct two distinct diagrams for the limits of Segre type of $R^a_{\ b}$ in 5–D. Collecting together the information of these limiting diagrams one finally obtains the limiting diagram for Segre types in 5–D, shown in Fig. 2.

In brief, to achieve this limiting diagram for the Segre classification of a second order symmetric two-tensor in 5–D we have essentially used the hereditary property (4.1) together with the limiting diagrams for the characteristic and minimal polynomial types, which are presented in [42].
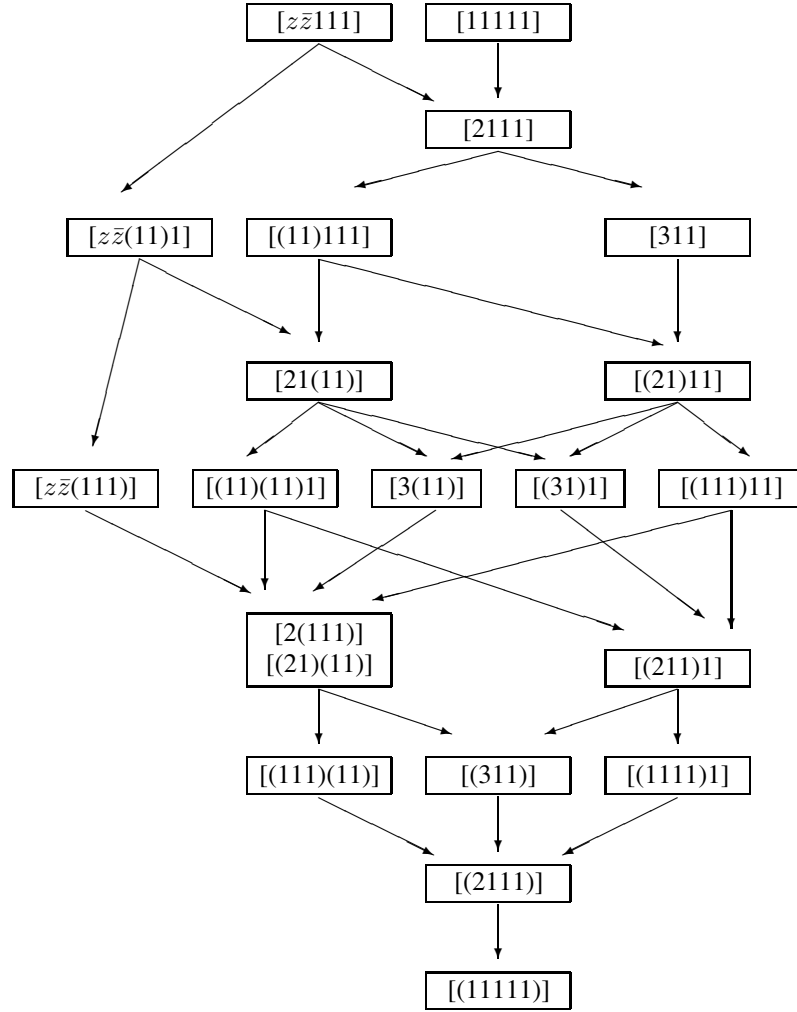
Figure 2. Diagram for the limits of the Segre types of R^a_b in 5–D Lorentzian spaces.

Improvements of the limiting diagram can still be tackled. A first would arise by taking into account the character of the eigenvectors. A second refinement of the limiting diagram can be made by devising a criterion for separating the Segre types $[2(111)]$ and $[(21)(11)]$, which have the same type for both characteristic and minimal polynomials. A third improvement might arise if besides the type of the characteristic and minimal polynomials one considers the values of their roots.

To close this section we note that although the coordinate-free technique for finding out limits of spacetimes in GR [39] have not yet been extended to 5–D spacetimes, the limiting diagram studied in this section will certainly be applicable to any coordinate-free approach to possible limits of non-vacuum space-times in 5–D.

5 Classification in n–D

In this section we shall discuss how one can obtain, by induction, the algebraic classification and the canonical forms of R^a_b on a n-dimensional ($n > 5$) Lorentzian space, with signature $(-+\cdots+)$, from its classification in 5–D.

By a procedure similar to that used in the theorem 1, it is easy to show that R^a_b defined on an n-dimensional ($n \geq 5$) Lorentzian space has at least one real space-like eigenvector.

The existence of this eigenvector can be used to reduce, by induction, the classification of symmetric two-tensors on n-dimensional ($n > 5$) spaces (and the corresponding canonical forms) to the classification on 5-dimensional spaces (and its canonical forms), thus recovering in a straightforward way the results of [6].

Indeed, with this real non-null n–D eigenvector ($\bar{\mathbf{v}}$, say) one can define another normalized eigenvector $\bar{\mathbf{u}}$, whose associated eigenvalue $\bar{\eta}$ is the same of $\bar{\mathbf{v}}$ in quite the same way $\mathbf{v}$ and $\mathbf{u}$ where defined in (3.11). Now, one can follow similar steps (and arguments) in n–D to those used in 5–D to go from (3.13) to (3.14). Now, introducing a semi-null basis $\mathcal{B}$ for the n–D space $T_p(M)$, consisting of 2 null vectors and $n - 2$ spacelike vectors,

$$\mathcal{B} = \{\mathbf{l}, \mathbf{m}, \mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(n-2)}\}, \qquad (5.1)$$

such that the only non-vanishing inner products are

$$\mathbf{l}.\mathbf{m} = \mathbf{x}^{(1)}.\mathbf{x}^{(1)} = \mathbf{x}^{(2)}.\mathbf{x}^{(2)} = \ldots = \mathbf{x}^{(n-2)}.\mathbf{x}^{(n-2)} = 1, \qquad (5.2)$$

and using the classification for R^a_b in 5–D obtained in section 3, it follows that the possible Segre types and the corresponding canonical forms for R_{ab} in n dimensions are given by

Segre type	Canonical form

$$[1,1\ldots1] \qquad R_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b + m_a m_b) + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)}$$
$$+ \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)} , \qquad (5.3)$$

$$[21\ldots1] \qquad R_{ab} = 2\rho_1 l_{(a}m_{b)} \pm l_a l_b + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)}$$
$$+ \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)} , \qquad (5.4)$$

$$[31\ldots1] \qquad R_{ab} = 2\rho_1 l_{(a}m_{b)} + 2l_{(a}x_{b)} + \rho_1 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)}$$
$$+ \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)} , \qquad (5.5)$$

$$[z\bar{z}11\ldots1] \qquad R_{ab} = 2\rho_1 l_{(a}m_{b)} + \rho_2(l_a l_b - m_a m_b) + \rho_3 x_a^{(1)} x_b^{(1)} + \rho_4 x_a^{(2)} x_b^{(2)}$$
$$+ \cdots + \rho_n x_a^{(n-2)} x_b^{(n-2)} , \qquad (5.6)$$

and the degeneracies thereof. Here the coefficients $\rho_1, \ldots, \rho_n$ are real scalars and $\rho_2 \neq 0$ in (5.6).

This classification of symmetric two-tensors in any n-dimensional spaces and their canonical forms are important in the context of n-dimensional brane-worlds as well as in the framework of 11–D supergravity and 10–D superstrings.

6 Further Results

In this section we shall briefly discuss some recent results [43] concerning the algebraic structure of a second order symmetric tensor R defined on a 5–D Lorentzian manifold M, which can be collected together in the following theorems:

Theorem 2 *Let M be a real 5-dimensional manifold endowed with a Lorentzian metric g of signature $(-++++)$. Let R^a_b be the mixed form of a second order symmetric tensor R defined at a point $p \in M$. Then*

(i) *R^a_b has a time-like eigenvector if and only if it is diagonalizable over $\mathbb{R}$ at p.*

(ii) *R^a_b has at least three real orthogonal independent eigenvectors at p, two of which (at least) are space-like.*

(iii) *R^a_b has all eigenvalues real at p and is not diagonalizable if and only if it has an unique null eigendirection at p.*

(iv) *If R^a_b has two linearly independent null eigenvectors at p then it is diagonalizable over $\mathbb{R}$ at p and the corresponding eigenvalues are real.*

Before stating the next theorem we recall that the r-dimensional ($r \geq 2$) subspaces of $T_p(M)$ can be classified according as they contain more than one, exactly one, or no null independent vectors, and they are respectively called time-like, null and space-like r-subspaces of $T_p(M)$. Space-like, null and time-like r-subspaces contain, respectively, only space-like vectors, null and space-like vectors, or all types of vectors.

Theorem 3 *Let M be a real 5-dimensional manifold endowed with a Lorentzian metric g of signature $(-++++)$. Let R^a_b be the mixed form of a second order symmetric tensor R defined at a point $p \in M$. Then*

(i) *There always exists a 2–D space-like subspace of $T_p(M)$ invariant under R^a_b.*

(ii) *If a non-null subspace $\mathcal{V}$ of $T_p(M)$ is invariant under R^a_b, then so is its orthogonal complement $\mathcal{V}^\perp$.*

(iii) *There always exists a 3–D time-like subspace of $T_p(M)$ invariant under R^a_b.*

(iv) *R^a_b admits a r-dimensional ($r = 2, 3, 4$) null invariant subspace $\mathcal{N}$ of $T_p(M)$ if and only if R^a_b has a null eigenvector, which lies in $\mathcal{N}$.*

The proofs can be gathered essentially from the canonical forms (3.15) – (3.18) and are not presented here for the sake of brevity, but the reader can find them in details in reference [43].

7 Concluding Remarks

Recent results on the expansion of the bulk geometry have raised some experimental and theoretical difficulties to 5–D brane-world models (for a clear indication of these problems see Damour *et al.* [44]). In particular, the massive gravitons (scalar-like polarization states) coupling to matter modify the usual general relativistic relation for interaction of matter and light, giving rise to a sharp discrepancy in the bending of light rays by the Sun, for example, so well explained by general relativity. Furthermore, in a recent paper [45] Durrer and Kocian have obtained a modification of Einstein quadrupole formula for the emission of gravity waves by a binary pulsar in the framework of 5–D brane-worlds. They have computed the induced change for the binary pulsar

PSR 1913+16 and shown that it amounts to about 20% in sharp contradiction with the current observations. Such observational problems challenge the brane-world scheme — a deeper understanding of the nature of the bulk, its dynamics, and how the embedding relates the bulk geometry to that of the brane, clearly is necessary.

The Segre classification, together with the limiting diagram in 5–D, and the extension of Geroch's limit could, in principle, offer ways of circumventing the current serious difficulties in 5–D brane-world. In fact, on the one hand it is well know that the Riemann tensor is decomposable into three irreducible parts, namely the Weyl tensor, the Ricci tensor and the Ricci scalar. On the other hand, it is known that the embedding is defined by the components of the Riemann tensor of the bulk (see, e.g., [46, 47]). Therefore one can potentially use the algebraic classification of the Weyl and Ricci parts of the 5–D curvature tensor to shed light into the above mentioned problem faced by in 5–D brane-world models. So, for example, the embedding conditions can in principle be used to relate Segre types in the 5–D bulk with Segre types of the 4–D brane-world (we note the limiting diagrams and Geroch's limiting theorem of the present paper hold for a fixed n). One could also find out, e.g., for which Segre and Petrov types (if any) associated to Riemann tensor of the bulk the above mentioned problems would not come about. Such an interesting research project does not seem to be straightforward, and is beyond the scope of the present review, though. We note in passing that this project would require not only the Segre classification but also the Petrov types of the Weyl tensor in 5–D, and to the best of our knowledge, the classification of Weyl tensor in five and higher dimensions has not yet been performed (note, however, section 2 of the recent article by De Smet [48]). We also emphasize that although recent interest in 5–D brane-world models has motivated this paper, our results apply to any second order symmetric tensor in the context of any locally Lorentzian geometrical theory.

To close this article we mention that it has sometimes been assumed, in the framework of brane-world cosmologies, that the source term ($\mathcal{T}_{ab}$ in 4–D) restricted to the brane is a mixture of ordinary matter and a minimally coupled scalar field [49, 50], where the gradient of the scalar field $\phi_\alpha \equiv \phi_{;\,\alpha}$ is a time-like vector. In these cases the scalar field can mimic a perfect fluid, i.e., it is of Segre type $[1, (111)]$. However, according to ref. [24], depending on the character of the gradient of the scalar field (and also on the character of the gradient of the corresponding potential) it can also mimic : (i) a null electromagnetic field and pure radiation (Segre type $[(211)]$, when ϕ^α is a null vector); (ii) a tachyon fluid (Segre type $[(1, 11)1]$, when ϕ^α is a space-like vector); and clearly (iii) a Λ term (Segre type $[(1, 111)]$, when the scalar field is a constant).

Acknowledgments

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