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A note on tensor fields in Hilbert spaces

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ABSTRACT

We discuss and extend to infinite dimensional Hilbert spaces a well-known tensoriality criterion for linear endomorphisms of the space of smooth vector fields in $\mathbb{R}^n$.

Key words: Tensor fields, Hilbert spaces.

1 INTRODUCTION

A basic fact in classical differential geometry, called the "Fundamental lemma of differential geometry" in Besse (1987), is that an $\mathbb{R}$-linear endomorphism of the space of vector fields in an open set of $\mathbb{R}^n$, is a tensor field (of type (1,1)) if and only if is linear with respect to functions. This is not any more true in infinite dimensions and the aim of this note is to give a contra-example and to introduce a class of endomorphism for which the criterion holds true. Some basic references for infinite dimensional differential geometry are Lang (1995) and Abraham et al. (1988).

2 TENSOR FIELDS IN HILBERT SPACES

Let $\mathbb{H}$ be a real Hilbert space, $\mathrm{B}(\mathbb{H})$ be the space of bounded linear endomorphisms of $\mathbb{H}$ and $\Omega \subset \mathbb{H}$ be an open set. We will denote by $\mathcal{H}(\Omega)$ the space of (smooth) vector fields in Ω, i.e., the space of smooth maps $\xi : \Omega \to \mathbb{H}$; by $\mathcal{F}(\Omega)$ we denote the algebra of smooth real valued functions in Ω. Then $\mathcal{H}(\Omega)$ is a real vector space and an $\mathcal{F}(\Omega)$-module in the obvious way.

We consider a map $A : \Omega \to \mathrm{B}(\mathbb{H})$ and, for $\xi \in \mathcal{H}(\Omega)$, we define a vector field $\tilde{A}(\xi)$ in Ω by setting $\tilde{A}(\xi)(x) = A(x) \cdot \xi(x)$. If $\tilde{A}(\xi)$ is smooth for all $\xi \in \mathcal{H}(\Omega)$ then we way that A is *weakly*

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smooth; in this case $\tilde{A}$ is a $\mathcal{F}(\Omega)$-linear endomorphism of $\mathcal{H}(\Omega)$. Clearly, if A is smooth then it is also weekly smooth and if $\dim(\mathbb{H}) < +\infty$ the converse holds.

Now we consider an arbitrary $\mathbb{R}$-linear map $\tilde{A} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$. Recall that:

1. $\tilde{A}$ is a *tensor field* (of type $(1,1)$) if there exist a weakly smooth map $A : \Omega \to \mathrm{B}(\mathbb{H})$ such that: $\tilde{A}(\xi)(x) = A(x) \cdot \xi(x)$, for all $x \in \Omega$.

2. $\tilde{A}$ is *punctual* if, for all $\xi \in \mathcal{H}(\Omega)$, $x \in \Omega$, $\xi(x) = 0$ implies $\tilde{A}(\xi)(x) = 0$.

3. $\tilde{A}$ is a *zero-order differential operator* if $\tilde{A}$ is $\mathcal{F}(\Omega)$-linear.

REMARK 1. The $\mathcal{F}(\Omega)$-linearity implies that $\tilde{A}$ is a *differential operator*, i.e., $\xi|_U = 0$ implies $\tilde{A}(\xi)|_U = 0$ for all $\xi \in \mathcal{H}(\Omega)$ and for every open subset $U \subset \Omega$.

Conditions (1) and (2) are obviously equivalent and any of them implies condition (3). It is well known and easy to prove that if $\dim(\mathbb{H}) = n < +\infty$ then conditions (1), (2) and (3) are all equivalent. We briefly recall the proof of the implication (3)$\Rightarrow$(2). Let $(e_i)_{i=1}^n$, $n = \dim(\mathbb{H}) < +\infty$, be a basis for $\mathbb{H}$ and, given $\xi \in \mathcal{H}(\Omega)$, write $\xi = \sum_{i=1}^n \xi_i e_i$, where each $\xi_i \in \mathcal{F}(\Omega)$. The $\mathcal{F}(\Omega)$-linearity of $\tilde{A}$ implies $\tilde{A}(\xi) = \sum_{i=1}^n \xi_i \tilde{A}(e_i)$ from which condition (2) follows.

The aim of this note is to discuss the relations between conditions (2) and (3) in the case that $\mathbb{H}$ is infinite-dimensional. We start by pointing out that such conditions are no longer equivalent as the following example shows.

EXAMPLE 2. Let $\lambda : \mathrm{B}(\mathbb{H}) \to \mathbb{H}$ be a continuous non zero linear map whose kernel contains the closed subspace of compact endomorphisms of $\mathbb{H}$. Set:

$$\tilde{A}(\xi)(x) = \lambda\big(\mathrm{d}\xi(x)\big), \quad \xi \in \mathcal{H}(\Omega),\ x \in \Omega,$$

where $\mathrm{d}\xi(x) \in \mathrm{B}(\mathbb{H})$ denotes the differential of the smooth map ξ at the point x. We claim that $\tilde{A}$ is an $\mathcal{F}(\Omega)$-linear operator. Namely, if $f \in \mathcal{F}(\Omega)$ and if $f(x) = 0$ for some $x \in \Omega$ then:

$$\mathrm{d}(f\xi)(x)v = \big[df(x)v\big]\xi(x).$$

Hence, $\mathrm{d}(f\xi)(x) \in \mathrm{B}(\mathbb{H})$ has 1-dimensional range and therefore it is a compact operator and $\tilde{A}(f\xi)(x) = 0$. Now, in general, if $f \in \mathcal{F}(\Omega)$ and $x \in \Omega$, we have:

$$\tilde{A}(f\xi)(x) = \tilde{A}\big[\big(f - f(x)\big)\xi + f(x)\xi\big](x) = f(x)\tilde{A}(\xi)(x),$$

which proves that $\tilde{A}$ is $\mathcal{F}(\Omega)$-linear. On the other hand $\tilde{A}$ cannot be punctual because if $\xi \in \mathrm{B}(\mathbb{H})$ is not in the kernel of λ then $\tilde{A}(\xi)(0)$ is not zero.

If we assume the continuity of the operator $\tilde{A}$ with respect to pointwise convergence in $\mathcal{H}(\Omega)$ then we can proceed in analogy with the finite dimensional case to prove the implication (3)$\Rightarrow$(2). Such assumption, however, is too strong since in all interesting cases directional derivatives appear in the expression for $\tilde{A}$. Assume, for simplicity, that $\mathbb{H}$ is separable (the general case follow substituting sequences for nets). Fix a Hilbert space basis $(e_i)_{i=1}^{+\infty}$ for $\mathbb{H}$ then, for $\xi \in \mathcal{H}(\Omega)$, we have $\xi = \sum_{i=1}^{+\infty} \xi_i e_i$, $\xi_i \in \mathcal{F}(\Omega)$, and this series converges pointwise together with all its directional derivatives. In fact, the *i-th coordinate operator* $\mathbb{H} \ni \xi \mapsto \xi_i = \langle \xi, e_i \rangle \in \mathbb{R}$ is linear and continuous and hence it commutes with directional derivative operators $\dfrac{\partial}{\partial v} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$, $v \in \mathbb{H}$.

The above considerations lead to the following:

DEFINITION 3. An operator $\tilde{A} : \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ is *weakly C^∞-continuous* if for every sequence $(\xi^k)_{k\in\mathbb{N}}$ in $\mathcal{H}(\Omega)$ converging pointwise with all its directional derivatives to $\xi \in \mathcal{H}(\Omega)$, the sequence $\big(\tilde{A}(\xi^k)(x)\big)_{k\in\mathbb{N}}$ converges to $\tilde{A}(\xi)(x)$ weakly in $\mathbb{H}$, for all $x \in \Omega$.

THEOREM 4. *If $\tilde{A}$ is weakly C^∞-continuous, then (3)$\Rightarrow$(2).*

PROOF. Let $\xi \in \mathcal{H}(\Omega)$, $x \in \Omega$ and assume $\xi(x) = 0$. We write $\xi = \sum_{i=1}^{+\infty} \xi_i e_i$, $\xi_i \in \mathcal{F}(\Omega)$, where $(e_i)_{i=1}^{+\infty}$ is a Hilbert basis for $\mathbb{H}$. By setting $\xi^k = \sum_{i=1}^{k} \xi_i e_i$ then, as observed above, $(\xi^k)_{k\in\mathbb{N}}$ converges pointwise to ξ together with all its directional derivatives. Since $\tilde{A}$ is weakly C^∞-continuous, $\big(\tilde{A}(\xi^k)(x)\big)_{k\in\mathbb{N}}$ converges weakly to $\tilde{A}(\xi)(x)$ in $\mathbb{H}$. Finally, by the $\mathcal{F}(\Omega)$-linearity of $\tilde{A}$:

$$\tilde{A}(\xi^k)(x) = \sum_{i=1}^{k} \xi_i(x)\tilde{A}(e_i)(x) = 0, \quad \text{for all } k \in \mathbb{N},$$

which implies $\tilde{A}(\xi)(x) = 0$ and concludes the proof. $\square$

We will now discuss the result above in an interesting case. We recall that a *linear connection* in $\mathbb{H}$ is an $\mathbb{R}$-bilinear map:

$$\nabla : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \ni (X, Y) \longmapsto \nabla_X Y \in \mathcal{H}(\Omega),$$

such that:

- ∇ is $\mathcal{F}(\Omega)$-linear in the first variable;
- for all $X, Y \in \mathcal{H}(\Omega)$, $f \in \mathcal{F}(\Omega)$, the identity:

  $$\nabla_X(fY) = X(f)Y + f\nabla_X Y,$$

  holds.

An easy modification of Example 2 shows that not all connections are punctual in the first argument. However, if we give a *Riemannian metric* in Ω i.e., a smooth map $g : \Omega \to \mathrm{B}(\mathbb{H})$ such that for all $x \in \Omega$, $g(x) : \mathbb{H} \to \mathbb{H}$ is self-adjoint and there exists a positive real-valued function K such that $\|g(x)v\| \geq K(x)\|v\|$, we have a special connection, the *Levi-Civita* connection, uniquely defined by the two extra conditions:

- $\nabla_X Y - \nabla_Y X = [X, Y] = dY(X) - dX(Y)$, for all $X, Y \in \mathcal{H}(\Omega)$;
- $Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$, for all $X, Y, Z \in \mathcal{H}(\Omega)$,

where $g(X, Y)(x) = \langle g(x)X, Y\rangle$. Observe that, for all $x \in \Omega$, $g(x)$ defines an inner product in $\mathbb{H}$ compatible with its topology. The Levi-Civita connection can be explicitly described by the *Koszul formula*:

$$\begin{aligned} 2g(\nabla_X Y, Z) = &Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &+ g\big([X, Y], Z\big) + g\big([Z, X], Y\big) - g\big([Y, Z], X\big). \end{aligned} \tag{1}$$

It follows from the properties of the Lie brackets, that the expression on the righthand side of (1) is $\mathcal{F}(\Omega)$-linear in X and in Z. Moreover, since such expression involves only partial derivatives, it is also weakly C^∞-continuous on X and Z. Hence, Theorem 4 implies that the righthand side of (1) is indeed punctual in X and Z, so that $\nabla_X Y$ is well-defined by (1) and it is punctual in X.

In general, given a connection ∇ we can write $\nabla_X Y = dY(X) + \Gamma(X, Y)$, where $\Gamma : \mathcal{H}(\Omega) \times \mathcal{H}(\Omega) \to \mathcal{H}(\Omega)$ is $\mathcal{F}(\Omega)$-bilinear. Classically, Γ is known as the *Christoffel symbol* of the connection. Using essentially the same argument as above, we conclude that the Christoffel symbol of the Levi-Civita connection is punctual in both arguments.

Also we can consider the *Riemannian curvature*:

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

where ∇ denotes the Levi-Civita connection. R is $\mathcal{F}(\Omega)$-linear in all arguments. Again, the fact that R is weakly C^∞-continuous implies, by Theorem 4, that R is punctual in all its variables.

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RESUMO

Discutimos e estendemos para espaços de Hilbert um critério de tensorialidade para endomorfismos do espaço dos campos vetoriais em $\mathbb{R}^n$.

Palavras-chave: campos tensoriais, espaços de Hilbert.

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