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A note on the connectedness locus of the families of polynomials $P_c(z) = z^n - cz^{n-j}$

CARLOS ARTEAGA[1] **and ALEXANDRE ALVES**[2]

[1] Departamento de Matemática, ICEX, Universidade Federal de Minas Gerais,
Av. Antonio Carlos, 6627, 31270-970 Belo Horizonte, MG, Brasil
[2] Departamento de Matemática, CCE, Universidade Federal de Viçosa,
Av. Peter Henry Rolfs, s/n, 36570-000 Viçosa, MG, Brasil

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ABSTRACT

Let j be a positive integer. For each integer $n > j$ we consider the connectedness locus $\mathcal{M}_n$ of the family of polynomials $P_c(z) = z^n - cz^{n-j}$, where c is a complex parameter. We prove that $\lim_{n\to\infty} \mathcal{M}_n = \mathbf{D}$ in the Hausdorff topology, where $\mathbf{D}$ is the unitary closed disk $\{c; |c| \leq 1\}$.

Key words: Julia set, connectedness locus, hyperbolic components, principal components.

1 INTRODUCTION

In (Milnor 2009), J. Milnor considers the complex 1-dimensional slice S_1 of the cubic polynomials that have a superatracting fixed point. He gives a detailed pictured of S_1 in dynamical terms. In (Roesch 2007), Roesch generalizes these results for families of polynomials of degree $n \geq 3$ having a critical fixed point of maximal multiplicity. This set of polynomials is described -modulo affine conjugacy- by the polynomials $P_c(z) = z^n - cz^{n-1}$. Roesch proved that the global pictures of the connectedness locus of this family of polynomials is a closed topological disk together with "limbs" sprouting off it at the cusps of Mandelbrot copies. In this note, we consider a positive integer j, and for each integer $n > j$, we consider the family of polynomials $P_c(z) = z^n - cz^{n-j}$, where c is a complex parameter. By definition, the **connectedness locus** $\mathcal{M}_n$ of this family of polynomials consists of all parameters c such that the Julia set of $P_c(z)$ is connected or equivalentely if the orbit of every critical point of $P_c(z)$ is bounded (see Carleson and Gamelin 1992). Since for all parameter $c; z = 0$ is a superattracting fixed point of $P_c(z)$, we deduce that $\mathcal{M}_n$ consists of all parameter c such that the orbit of every non-zero critical point of $P_c(z)$ is bounded. We also consider the space of non-empty compacts subsets of the plane eqquiped with the Hausdorff distance (see Douady 1994).We obtain the following result about the size of $\mathcal{M}_n$.

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Correspondence to: Carlos Arteaga
E-mail: dcam@mat.ufmg.br

THEOREM A. *$\mathcal{M}_n$ is a non-empty compact subset of the plane and*

$$\lim_{n\to\infty}(\mathcal{M}_n) = \mathbf{D},$$

in the Hausdorff topology, where $\mathbf{D}$ *is the unitary closed disk* $\{c;\ |c| \leq 1\}$.

2 PROOF OF THEOREM A

The proof of the Theorem is based in the following results.

LEMMA 2.1. For $n > 3j$, *the closed unitary disk* $\mathbf{D}$ *is contained in* $\mathcal{M}_n$.

PROOF. Let $c \in \mathbf{D}$ and let $k = \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}}\left(\frac{j}{n-j}\right)$. Since $n > 3j$, we have that $\frac{j}{n-j} < \frac{1}{2}$, so $k < \frac{1}{2}$. Let z_c be a non-zero critical point of $P_c(z)$. Then, $z_c^j = \frac{n-j}{n}c$, and this implies that

$$P_c(z_c) = z_c^n - cz_c^{n-j} = z_c^n - \left(\frac{n}{n-j}\right)z_c^n = -\left(\frac{j}{n-j}\right)z_c^n.$$

This and the fact that

$$|z_c|^{n-1} = \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}}|c|^{\frac{n-1}{j}}$$

imply that

$$|P_c(z_c)| = \frac{j}{n-j}\left(\frac{n-j}{n}\right)^{\frac{n-1}{j}}|z_c| = k|c|^{\frac{n-1}{j}}|z_c|.$$

Hence, since $|c| < 1$, $P_c(z_c)| \leq k\,|z_c|$.

By induction, suppose that $|P_c^q(z_c)\,| \leq k^q\,|z_c|$. Then,

$$\begin{aligned}
|P_c^{q+1}(z_c)| &= |P_c^q(z_c)|^{n-j}\,|(P_c^q(z_c))^j - c| = |P_c^q(z_c)|^{n-j}\,|(P_c^q(z_c))^j - \frac{n}{n-j}\,z_c^j| \\
&= |P_c^q(z_c)|^{n-j}\,|z_c|^j\,\left|\left(\frac{P_c^q(z_c)}{z_c}\right)^j - \frac{n}{n-j}\right| \leq k^{q(n-j)}\,|z_c|^n\left(k^{qj} + \frac{n}{n-j}\right) \\
&\leq k^{q\,(n-j-1)-1}\left(k + \frac{n}{n-j}\right)k^{q+1}|z_c|.
\end{aligned}$$

where the last inequality is true because $|z_c| < 1$ and $k < 1$.

On the other hand, since $n > 3j$, $\frac{n}{n-j} < \frac{3}{2}$ and $q\,(n-j-1) - 1 > 1$. Thus,

$$k^{q\,(n-j-1)-1}\left(k + \frac{n}{n-j}\right) < k\left(k + \frac{3}{2}\right) < \frac{1}{2}\left(\frac{1}{2} + \frac{3}{2}\right) = 1.$$

Combinated with the estimate above, this gives $|P_c^{q+1}(z_c)\,| \leq k^{q+1}\,|z_c|$. Hence, $|P_c^q(z_c)| \leq k^q\,|z_c|$ for all positive integer q. Since $k < 1$, we deduce that the orbit $\{P_c^q(z_c)\}$ is bounded and Lemma 2.1 is proved.

LEMMA 2.2. *If* $n > j$, *then* $\mathcal{M}_n$ *is a subset of the disk* $\left\{c;\ |c| \leq \left(\frac{n-j}{j}\right)^{\frac{j}{n-1}}\left(\frac{n}{n-j}\right)^2\right\}$.

PROOF. Let $|c| > \left(\frac{n-j}{j}\right)^{\frac{j}{n-1}} \left(\frac{n}{n-j}\right)^2$. By definition of $\mathcal{M}_n$, we have that, in order to prove Lemma 2.2, it is sufficient to prove that, for each non-zero critical point z_c of $P_c(z) = z^n - cz^{n-j}$, the orbit $\{P_c^q(z_c)\}$ is not bounded.

Let $k = \frac{j}{n-j} |z_c|^{n-1}$. We claim that $k > \left(\frac{n}{n-j}\right)^{\frac{n-1}{j}}$ and hence $k > 1$.

In fact, since $z_c^j = \frac{n-j}{n} c$,

$$k = \frac{j}{n-j} \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}} |c|^{\frac{n-1}{j}} > \left(\frac{j}{n-j}\right) \left(\frac{n-j}{n}\right)^{\frac{n-1}{j}} \left(\frac{n-j}{j}\right) \left(\frac{n}{n-j}\right)^{\frac{2(n-1)}{j}} > \left(\frac{n}{n-j}\right)^{\frac{n-1}{j}},$$

and the claim is proved.

Now, we have that

$$|P_c(z_c)| = |z_c^n - cz_c^{n-j}| = |z_c^n - \frac{n}{n-j} z_c^n| = \frac{j}{n-j} |z_c|^n = k\, |z_c|$$

By induction, suppose that $|P_c^q(z_c)| \geq k^q\, |z_c|$. Then,

$$\begin{aligned} |P_c^{q+1}(z_c)| &= |P_c^q(z_c)|^{n-j}\, |P_c^q(z_c))^j - c| = |P_c^q(z_c)|^{n-j}\, |z_c|^j \left| \left(\frac{P_c^q(z_c)}{z_c}\right)^j - \frac{n}{n-j} \right| \\ &\geq k^{q(n-j)} |z_c|^n \left(k^{qj} - \frac{n}{n-j}\right) = k^{q(n-j)} k \left(\frac{n-j}{j}\right)\left(\frac{n}{n-j}\right) \left(\frac{n-j}{n} k^{qj} - 1\right) |z_c| \\ &\geq \frac{n}{j} \left(\frac{n-j}{n} k^{qj} - 1\right) k^{q+1} |z_c| \geq \frac{n}{j} \left(\left(\frac{n}{n-j}\right)^{q(n-1)-1} - 1\right) k^{q+1} |z_c|. \end{aligned}$$

where the last inequality follows from the Claim above.

On the other hand, let $s = q(n-1) - 1$. Then, $s > 1$ and

$$\begin{aligned} \frac{n}{j}\left(\left(\frac{n}{n-j}\right)^s - 1\right) &= \frac{n}{j}\left(\frac{n}{n-j} - 1\right)\left(\left(\frac{n}{n-j}\right)^{s-1} + \cdots + 1\right) \\ &= \frac{n}{n-j}\left(\left(\frac{n}{n-j}\right)^{s-1} + \cdots + 1\right) > 1. \end{aligned}$$

Combinated with the estimates above, this gives $|P_c^{q+1}(z_c)| \geq k^{q+1}\, |z_c|$. Hence, $|P_c^q(z_c)| > k^q\, |z_c|$ for all positive integer q. Since $k > 1$, we conclude that, for each critical point z_c of $P_c(z)$, the orbit $\{P_c^q(z_c)\}$is not bounded, and Lemma 2.2 is proved.

Now, we prove Theorem A. By Lemma 2.2, $\mathcal{M}_n$ is bounded.

Let $J = \left(\frac{n-j}{j}\right)^{\frac{j}{n-1}} \left(\frac{n}{n-j}\right)^2$ and let L be a positive integer such that $L^j - J > 1$. Suppose by contradiction that $\mathcal{M}_n$ is not closed. Then, there exists d in the boundary $\partial\mathcal{M}_n$ of $\mathcal{M}_n$ such that the orbit$\{P_d^l(z_d)\}$ is not bounded for some non-zero critical point z_d of $P_d(z)$. Hence, there exists a positive integer

q such that $|P_d^q(z_d)| > L$. Since $z_d^j = \frac{n-j}{n} d$, we can choose a local branch of $F(c) = \left(\frac{n-j}{n} c\right)^{\frac{1}{j}}$ in a neighborhood V of d such that $|P_c^q(z_c)| > L$, for all $c \in V$. Since $d \in \partial \mathcal{M}_n$, there exists $c \in \mathcal{M}_n \cap V$ such that $|P_c^q(z_c)| > L$. By Lemma 2.2, $|c| < j$. Let $\omega = P_c^q(z_c)$. Then,

$$|\omega|^j - |c| > L^j - J > 1,$$

thus,

$$|P_c(\omega)| = |\omega^{n-j}||\omega^j - c| > L.$$

By induction, suppose that $|P_c^m(\omega)| > L^m$. Then, $|P_c^m(\omega)|^j - |c| > L^{mj} - J > L$. It follows that,

$$|P_c^{m+1}(\omega)| = |P_c^m(\omega)|^{n-j} |(P_c^m(\omega))^j - c| > L^{m(n-j)} L > L^{m+1}.$$

Hence, the orbit $\{P_c^l(z_c)\}$ is not bounded. This is a contradiction because $c \in \mathcal{M}_n$. Therefore, $\mathcal{M}_n$ is closed, so it is compact. Now, Lemmas 2.1 and 2.2 and the fact that $\lim_{n\to\infty} \left(\frac{n-j}{j}\right)^{\frac{j}{n-1}} \left(\frac{n}{n-j}\right)^2 = 1$ imply that $\lim_{n\to\infty} \mathcal{M}_n = \mathbf{D}$ in the Hausdorff topology, and Theorem A is proved.

RESUMO

Seja j um inteiro positivo. Para cada inteiro $n > j$, consideramos o locus conexo $\mathcal{M}_n$ da família de polinômios $P_c(z) = z^n - cz^{n-j}$, onde c é um parâmetro complexo. Provamos que $\lim_{n\to\infty} \mathcal{M}_n = \mathbf{D}$ na topologia de Hausdorff; onde $\mathbf{D}$ é o disco unitário $\{c; |c| \leq 1\}$.

Palavras-chave: Conjunto de Julia, locus conexo, componentes hiperbólicas, componente principal.

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